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# ADAPTIVE DENSITY ESTIMATION FOR GENERAL ARCH MODELS

F. COMTE<sup>\*,1</sup>, J. DEDECKER<sup>2</sup>, AND M. L. TAUPIN<sup>3</sup>

**ABSTRACT.** We consider a model  $Y_t = \sigma_t \eta_t$  in which  $(\sigma_t)$  is not independent of the noise process  $(\eta_t)$ , but  $\sigma_t$  is independent of  $\eta_t$  for each  $t$ . We assume that  $(\sigma_t)$  is stationary and we propose an adaptive estimator of the density of  $\ln(\sigma_t^2)$  based on the observations  $Y_t$ . Under various dependence structures, the rates of this nonparametric estimator coincide with the minimax rates obtained in the i.i.d. case when  $(\sigma_t)$  and  $(\eta_t)$  are independent, in all cases where these minimax rates are known. The results apply to various linear and non linear ARCH processes.

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## 1. INTRODUCTION

In this paper, we consider the following general ARCH-type model:  $((Y_t, \sigma_t))_{t \geq 0}$  is a strictly stationary sequence of  $\mathbb{R} \times \mathbb{R}^+$ -valued random variables, satisfying the equation

$$(1.1) \quad Y_t = \sigma_t \eta_t$$

where  $(\eta_t)_{t \in \mathbb{Z}}$  is a sequence of independent and identically distributed (i.i.d.) random variables with mean zero and finite variance, and for each  $t \geq 0$ , the random vector  $(\sigma_i, \eta_{i-1})_{0 \leq i \leq t}$  is independent of the sequence  $(\eta_i)_{i \geq t}$ .

The model is classically re-written *via* a logarithmic transformation:

$$(1.2) \quad Z_t = X_t + \varepsilon_t,$$

where  $Z_t = \ln(Y_t^2)$ ,  $X_t = \ln(\sigma_t^2)$  and  $\varepsilon_t = \ln(\eta_t^2)$ . In the context derived from the model (1.1),  $X_t$  and  $\varepsilon_t$  are independent for a given  $t$ , whereas the processes  $(X_t)_{t \geq 0}$  and  $(\varepsilon_t)_{t \in \mathbb{Z}}$  are not independent.

Our aim is the adaptive estimation of  $g$ , the common distribution of the unobserved variables  $X_t = \ln(\sigma_t^2)$ , when the density  $f_\varepsilon$  of  $\varepsilon_t = \ln(\eta_t^2)$  is known. More precisely we shall build an estimator of  $g$  without any prior knowledge on its smoothness, using the observations  $Z_t = \ln(Y_t^2)_t$  and the knowledge of the convolution kernel  $f_\varepsilon$ . Since  $X_t$  and  $\varepsilon_t$  are independent for each  $t$ , the common density  $f_Z$  of the  $Z_t$ 's is given by the convolution equation  $f_Z = g * f_\varepsilon$ .

In many papers dealing with ARCH models,  $\varepsilon_t$  is assumed to be Gaussian or the log of a squared Gaussian (when  $\eta_t$  is Gaussian, see van Es *et al.* (2005) or in slightly different

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contexts van Es *et al.* (2003), Comte and Genon-Catalot (2005)). Our setting is more general since we consider various type of error densities. More precisely, we assume that  $f_\varepsilon$  belongs to some class of smooth functions described below: there exist nonnegative numbers  $\kappa_0$ ,  $\gamma$ ,  $\mu$ , and  $\delta$  such that the fourier transform  $f_\varepsilon^*$  of  $f_\varepsilon$  satisfies

$$(1.3) \quad \kappa_0(x^2 + 1)^{-\gamma/2} \exp\{-\mu|x|^\delta\} \leq |f_\varepsilon^*(x)| \leq \kappa_0'(x^2 + 1)^{-\gamma/2} \exp\{-\mu|x|^\delta\}.$$

Since  $f_\varepsilon$  is known, the constants  $\mu, \delta, \kappa_0$ , and  $\gamma$  defined in (1.3) are known. When  $\delta = 0$  in (1.3), the errors are called “ordinary smooth” errors. When  $\mu > 0$  and  $\delta > 0$ , they are called “super smooth”. The standard examples for super smooth densities are Gaussian or Cauchy distributions (super smooth of order  $\gamma = 0, \delta = 2$  and  $\gamma = 0, \delta = 1$  respectively). When  $\varepsilon_t = \ln(\eta_t^2)$  with  $\eta_t \sim \mathcal{N}(0, 1)$  as in van Es *et al.* (2003, 2005), then  $\varepsilon_t$  is super-smooth with  $\delta = 1, \gamma = 0$  and  $\mu = \pi/2$ . An example of ordinary smooth density is the Laplace distribution, for which  $\delta = \mu = 0$  and  $\gamma = 2$ .

In density deconvolution of i.i.d variables the  $X_t$ 's and the  $\varepsilon_t$ 's are i.i.d. and the sequences  $(X_t)_{t \geq 0}$  and  $(\varepsilon_t)_{t \in \mathbb{Z}}$  are independent (for short we shall refer to this case as the i.i.d. case). In the setting of Model (1.2), the classical assumptions of independence between the processes  $(X_t)_{t \geq 0}$  and  $(\varepsilon_t)_{t \in \mathbb{Z}}$  are no longer satisfied and the tools for deconvolution have to be revisited.

As in density deconvolution for i.i.d. variables, the slowest rates of convergence for estimating  $g$  are obtained for super smooth error densities. For instance, in the i.i.d case, when  $\varepsilon_t$  is Gaussian or the log of a squared Gaussian and  $g$  belongs to some Sobolev class, the minimax rates are negative powers of  $\ln(n)$  (see Fan (1991)). Nevertheless, it has been noticed by several authors (see Pensky and Vidakovic (1999), Butucea (2004), Butucea and Tsybakov (2005), Comte *et al.* (2006)) that the rates are improved if  $g$  has stronger smoothness properties. So, we describe the smoothness properties of  $g$  by the set

$$(1.4) \quad \mathcal{S}_{s,r,b}(C_1) = \left\{ \psi \text{ such that } \int_{-\infty}^{+\infty} |\psi^*(x)|^2 (x^2 + 1)^s \exp\{2b|x|^r\} dx \leq C_1 \right\}$$

for  $s, r, b$  unknown non negative numbers. When  $r = 0$ , the class  $\mathcal{S}_{s,r,b}(C_1)$  corresponds to a Sobolev ball. When  $r > 0, b > 0$  functions belonging to  $\mathcal{S}_{s,r,b}(C_1)$  are infinitely many times differentiable.

Our estimator of  $g$  is constructed by minimizing an appropriate penalized contrast function only depending on the observations and on  $f_\varepsilon$ . It is chosen in a purely data-driven way among a collection of non-adaptive estimators. We start by the study of those non-adaptive estimators and show that their mean integrated squared error (MISE) has the same order as in the i.i.d. case. In particular they reach the minimax rates of the i.i.d. case in all cases where they are known (see Fan (1991), Butucea (2004) and Butucea and Tsybakov (2005)). Next we prove that the MISE of our adaptive estimator is of the same order as the MISE of the best non-adaptive estimator, up to some possible negligible logarithmic loss in one case.

In their 2005 paper, van Es *et al.* (2005) have considered the case where  $\eta_t$  is Gaussian, the density  $g$  of  $X_t$  is twice differentiable, and the process  $(Z_t, X_t)$  is  $\alpha$ -mixing. Here we consider various types of error density, and we do not make any assumption on the smoothness of  $g$ : this is the advantage of the adaptive procedure. We shall consider two types of dependence properties, which are satisfied by many ARCH processes. First

we shall use the classical  $\beta$ -mixing properties of general ARCH models, as recalled in Doukhan (1994) and described in more details in Carrasco and Chen (2002). But we also illustrate that new recent coefficients can be used in our context, which allow an easy characterization of the dependence properties in function of the parameters of the models. Those new dependence coefficients, recently defined and studied in Dedecker and Prieur (2005), are interesting and powerful because they require much lighter conditions on the models. Such ideas have been popularized by Ango Nzé and Doukhan (2004) and Doukhan *et al.* (2006). For instance, these coefficients allow to deal with the general ARCH( $\infty$ ) processes defined by Giraitis *et al.* (2000).

The paper is organized as follows. Many examples are described in Section 2, together with their dependence properties. The estimator is defined in Section 3. The MISE bounds are given in Section 4, and the proofs are given in Section 5.

## 2. THE MODEL AND ITS DEPENDENCE PROPERTIES

**2.1. Models and examples.** A particular case of model (1.1) is

$$(2.1) \quad Y_t = \sigma_t \eta_t, \text{ with } \sigma_t = f(\eta_{t-1}, \eta_{t-2}, \dots)$$

for some measurable function  $f$ . Another important case is

$$(2.2) \quad Y_t = \sigma_t \eta_t, \text{ with } \sigma_t = f(\sigma_{t-1}, \eta_{t-1}) \text{ and } \sigma_0 \text{ independent of } (\eta_t)_{t \geq 0},$$

that is  $\sigma_t$  is a stationary Markov chain.

We begin with models satisfying a recursive equation, whose stationary solution satisfies (2.1). The original ARCH model as introduced by Engle (1982) was given by

$$(2.3) \quad Y_t = \sqrt{a + bY_{t-1}^2} \eta_t, \quad a \geq 0, b \geq 0$$

It has been generalized by Bollerslev (1986) with the class of GARCH( $p, q$ ) models defined by  $Y_t = \sigma_t \eta_t$  and

$$(2.4) \quad \sigma_t^2 = a + \sum_{i=1}^p a_i Y_{t-i}^2 + \sum_{j=1}^q b_j \sigma_{t-j}^2$$

where the coefficients  $a, a_i, i = 1, \dots, p$  and  $b_j, j = 1, \dots, q$  are all positive real numbers. Those processes were studied from the point of view of existence and stationarity of solutions by Bougerol and Picard (1992a, 1992b) and Ango Nzé (1992). Under the condition  $\sum_{i=1}^p a_i + \sum_{j=1}^q b_j < 1$ , this model has a unique stationary solution of the form (2.1).

Many extensions have been proposed since then. A general linear example of model is given by the ARCH( $\infty$ ) model described by Giraitis *et al.* (2000):

$$(2.5) \quad \sigma_t^2 = a + \sum_{j=1}^{\infty} a_j Y_{t-j}^2,$$

where  $a \geq 0$  and  $a_j \geq 0$ . Again if  $\sum_{j \geq 1} a_j < 1$ , then there exists a unique strictly stationary solution to (2.5) of the form (2.1).

For the models satisfying (2.2), let us cite first the so-called augmented GARCH(1,1) models introduced by Duan (1997):

$$(2.6) \quad \Lambda(\sigma_t^2) = c(\eta_{t-1})\Lambda(\sigma_{t-1}^2) + h(\eta_{t-1}),$$

where  $\Lambda$  is an increasing and continuous function on  $\mathbb{R}^+$ . We refer to Duan (1997) for numerous examples of more standard models belonging to this class. There exists a stationary solution to (2.6), provided  $c$  satisfies the condition  $A_2^*$  given in Carrasco and Chen (2002) (this condition is satisfied as soon as  $\mathbb{E}(|c(\eta_0)|^s) < 1$  and  $\mathbb{E}(|h(\eta_0)|^s) < \infty$  for integer  $s \geq 1$ , see the condition  $A_2$  of the same paper). An example of the model (2.6) is the threshold ARCH model (see Zakoïan (1993)):

$$(2.7) \quad \sigma_t = a + b\sigma_{t-1}\eta_{t-1}\mathbf{1}_{\{\eta_{t-1}>0\}} - c\sigma_{t-1}\eta_{t-1}\mathbf{1}_{\{\eta_{t-1}<0\}}, \quad a, b, c > 0$$

for which  $c(\eta_{t-1}) = b\eta_{t-1}\mathbf{1}_{\{\eta_{t-1}>0\}} - c\eta_{t-1}\mathbf{1}_{\{\eta_{t-1}<0\}}$  and  $h = a$ . In particular, the condition for the stationarity is satisfied as soon as  $b \vee c < 1$ .

Other models satisfying (2.2) are the non linear ARCH models (see Doukhan (1994), p. 106-107), for which:

$$(2.8) \quad \sigma_t = f(\sigma_{t-1}\eta_{t-1}).$$

There exists a stationary solution to (2.8) provided that the density of  $\eta_0$  is positive on a neighborhood of 0 and  $\limsup_{|x| \rightarrow \infty} |f(x)/x| < 1$ .

In the next section, we define the dependence coefficients that we shall use in this paper, and we give the dependence properties of the models (2.3)-(2.8) in terms of these coefficients.

**2.2. Measures of dependence.** Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space. Let  $W$  be a random vector with values in a Banach space  $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ , and let  $\mathcal{M}$  be a  $\sigma$ -algebra of  $\mathcal{A}$ . Let  $\mathbb{P}_{W|\mathcal{M}}$  be a conditional distribution of  $W$  given  $\mathcal{M}$ , and let  $P_W$  be the distribution of  $W$ . Let  $\mathcal{B}(\mathbb{B})$  be the Borel  $\sigma$ -algebra on  $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ , and let  $\Lambda_1(\mathbb{B})$  be the set of 1-Lipschitz functions from  $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$  to  $\mathbb{R}$ . Define now

$$\beta(\mathcal{M}, \sigma(W)) = \mathbb{E} \left( \sup_{A \in \mathcal{B}(\mathcal{X})} |\mathbb{P}_{W|\mathcal{M}}(A) - \mathbb{P}_W(A)| \right),$$

$$\text{and if } \mathbb{E}(\|W\|_{\mathbb{B}}) < \infty, \quad \tau(\mathcal{M}, W) = \mathbb{E} \left( \sup_{f \in \Lambda_1(\mathbb{B})} |\mathbb{P}_{W|\mathcal{M}}(f) - \mathbb{P}_W(f)| \right).$$

The coefficient  $\beta(\mathcal{M}, \sigma(W))$  is the usual mixing coefficient, introduced by Rozanov and Volkonskii (1960). The coefficient  $\tau(\mathcal{M}, W)$  has been introduced by Dedecker and Prieur (2005).

Let  $(W_t)_{t \geq 0}$  be a strictly stationary sequence of  $\mathbb{R}^2$ -valued random variables. On  $\mathbb{R}^2$ , we put the norm  $\|x - y\|_{\mathbb{R}^2} = |x_1 - y_1| + |x_2 - y_2|$ . For any  $k \geq 0$ , define the coefficients

$$(2.9) \quad \beta_1(k) = \beta(\sigma(W_0), \sigma(W_k)), \quad \text{and if } \mathbb{E}(\|W_0\|_{\mathbb{R}^2}) < \infty, \quad \tau_1(k) = \tau(\sigma(W_0), W_k).$$

On  $(\mathbb{R}^2)^l$ , we put the norm  $\|x - y\|_{(\mathbb{R}^2)^l} = l^{-1}(\|x_1 - y_1\|_{\mathbb{R}^2} + \dots + \|x_l - y_l\|_{\mathbb{R}^2})$ . Let  $\mathcal{M}_i = \sigma(W_k, 0 \leq k \leq i)$ . The coefficients  $\beta_{\infty}(k)$  and  $\tau_{\infty}(k)$  are defined by

$$(2.10) \quad \beta_{\infty}(k) = \sup_{i \geq 0} \sup_{l \geq 1} \{ \beta(\mathcal{M}_i, \sigma(W_{i_1}, \dots, W_{i_l})), i + k \leq i_1 < \dots < i_l \},$$

and if  $\mathbb{E}(\|W_1\|_{\mathbb{R}^2}) < \infty$ ,

$$(2.11) \quad \tau_{\infty}(k) = \sup_{i \geq 0} \sup_{l \geq 1} \{ \tau(\mathcal{M}_i, (W_{i_1}, \dots, W_{i_l})), i + k \leq i_1 < \dots < i_l \}.$$

We say that the process  $(W_t)_{t \geq 0}$  is  $\beta$ -mixing (resp.  $\tau$ -dependent) if the coefficients  $\beta_\infty(k)$  (resp.  $\tau_\infty(k)$ ) tend to zero as  $k$  tends to infinity. We say that it is geometrically  $\beta$ -mixing (resp.  $\tau$ -dependent), if there exist  $a > 1$  and  $C > 0$  such that  $\beta_\infty(k) \leq Ca^k$  (resp.  $\tau_\infty(k) \leq Ca^k$ ) for all  $k \geq 1$ .

We now recall the coupling properties associated with the dependency coefficients. Assume that  $\Omega$  is rich enough, which means that there exists  $U$  uniformly distributed over  $[0, 1]$  and independent of  $\mathcal{M} \vee \sigma(W)$ . There exist two  $\mathcal{M} \vee \sigma(U) \vee \sigma(W)$ -measurable random variables  $W_1^*$  and  $W_2^*$  distributed as  $W$  and independent of  $\mathcal{M}$  such that

$$(2.12) \quad \beta(\mathcal{M}, \sigma(W)) = \mathbb{P}(W \neq W_1^*) \quad \text{and} \quad \tau(\mathcal{M}, W) = \mathbb{E}(\|W - W_2^*\|_{\mathbb{B}}).$$

The first equality in (2.12) is due to Berbee (1979), and the second one has been established in Dedecker and Priour (2005), Section 7.1.

As consequences of the coupling properties (2.12), we have the following covariance inequalities. Let  $\|\cdot\|_{\infty, \mathbb{P}}$  be the  $\mathbb{L}^\infty(\Omega, \mathbb{P})$ -norm. For two measurable functions  $f, h$  from  $\mathbb{R}$  to  $\mathbb{C}$ , we have

$$(2.13) \quad |\text{Cov}(f(Y), h(X))| \leq 2\|f(Y)\|_{\infty, \mathbb{P}}\|h(X)\|_{\infty, \mathbb{P}}\beta(\sigma(X), \sigma(Y)).$$

Moreover, if  $\text{Lip}(h)$  is the Lipschitz coefficient of  $h$ ,

$$(2.14) \quad |\text{Cov}(f(Y), h(X))| \leq \|f(Y)\|_{\infty, \mathbb{P}}\text{Lip}(h)\tau(\sigma(Y), X).$$

Thus, using that  $t \rightarrow e^{ixt}$  is  $|x|$ -Lipschitz, we obtain the bounds

$$(2.15) \quad |\text{Cov}(e^{ixZ_1}, e^{ixX_k})| \leq 2\beta_1(k-1) \quad \text{and} \quad |\text{Cov}(e^{ixZ_1}, e^{ixX_k})| \leq |x|\tau_1(k-1).$$

**2.3. Application to ARCH models.** For the models (1.1) and (1.2), the  $\beta$ -mixing coefficients of the process

$$(2.16) \quad (W_t)_{t \in \mathbb{Z}} = ((Z_t, X_t))_{t \in \mathbb{Z}}$$

are smaller than that of  $((Y_t, \sigma_t))_{t \in \mathbb{Z}}$  (because of the inclusion of  $\sigma$ -algebras). If we assume that in all cases the  $\eta_t$ 's are centered with unit variance and admit a density with respect to the Lebesgue measure, then

- The process  $((Y_t, \sigma_t))_{t \in \mathbb{Z}}$  defined by Model (2.3) is geometrically  $\beta$ -mixing as soon as  $0 < b < 1$ .
- The process  $((Y_t, \sigma_t))_{t \in \mathbb{Z}}$  defined by Model (2.4) is geometrically  $\beta$ -mixing, as soon as  $\sum_{i=1}^p a_i + \sum_{j=1}^q b_j < 1$  (see Carrasco and Chen (2000, 2002)).
- The process  $((Y_t, \sigma_t))_{t \in \mathbb{Z}}$  defined by Model (2.6) is geometrically  $\beta$ -mixing as soon as: the density of  $\eta_0$  is positive on an open set containing 0;  $c$  and  $h$  are polynomial functions; there exists an integer  $s \geq 1$  such that  $|c(0)| < 1$ ,  $\mathbb{E}(|c(\eta_0)|^s) < 1$ , and  $\mathbb{E}(|h(\eta_0)|^s) < \infty$ . See Proposition 5 in Carrasco and Chen (2002).
- The process  $((Y_t, \sigma_t))_{t \in \mathbb{Z}}$  defined by Model (2.7) is geometrically  $\beta$ -mixing as soon as  $0 < b \vee c < 1$ .
- The process  $((Y_t, \sigma_t))_{t \in \mathbb{Z}}$  defined by Model (2.8) is geometrically  $\beta$ -mixing as soon as the density of  $\eta_0$  is positive on a neighborhood of 0 and  $\limsup_{|x| \rightarrow +\infty} |f(x)/x| < 1$  (see Doukhan (1994), Proposition 6 page 107).

Note that some other extensions to nonlinear models having stationarity and dependency properties can be found in Lee and Shin (2005).

Concerning the  $\tau$ -dependence, here is a general method to handle the models (2.1) and (2.2). The following Proposition will be proved in appendix (see Ango Nzé and Doukhan (2004) and Doukhan *et al.* (2006) for related results).

**Proposition 2.1.** *Let  $Y_t$  and  $\sigma_t$  satisfy either (2.1) or (2.2). For Model (2.1), let  $(\eta'_t)_{t \in \mathbb{Z}}$  be an independent copy of  $(\eta_t)_{t \in \mathbb{Z}}$ , and for  $t > 0$ , let  $\sigma_t^* = f(\eta_{t-1}, \dots, \eta_1, \eta'_0, \eta'_{-1}, \dots)$ . For Model (2.2), let  $\sigma_0^*$  be a copy of  $\sigma_0$  independent of  $(\sigma_0, \eta_t)_{t \in \mathbb{Z}}$ , and for  $t > 0$  let  $\sigma_t^* = f(\sigma_{t-1}^*, \eta_{t-1})$ . Let  $\delta_n$  be a non increasing sequence such that*

$$(2.17) \quad 2\mathbb{E}(|\sigma_n^2 - (\sigma_n^*)^2|) \leq \delta_n.$$

Then

- (1) *The process  $((Y_t^2, \sigma_t^2))_{t \geq 0}$  is  $\tau$ -dependent with  $\tau_\infty(n) \leq \delta_n$ .*
- (2) *Assume that  $Y_0^2, \sigma_0^2$  have densities satisfying  $\max(f_{\sigma^2}(x), f_{Y^2}(x)) \leq C|\ln(x)|^\alpha x^{-\rho}$  in a neighborhood of 0, for some  $\alpha \geq 0$  and  $0 \leq \rho < 1$ . The process  $((X_t, Z_t))_{t \geq 0}$  is  $\tau$ -dependent with  $\tau_\infty(n) = O((\delta_n)^{(1-\rho)/(2-\rho)} |\ln(\delta_n)|^{(1+\alpha)/(2-\rho)})$ .*

Consider Model (2.5), and assume that  $c = \sum_{j \geq 1} a_j < 1$ . Let then  $((Y_t, \sigma_t))_{t \in \mathbb{Z}}$  be the unique strictly stationary solution of the form (2.1). Then (2.17) holds with

$$\delta_n = O\left(\inf_{1 \leq k \leq n} \left\{ c^{n/k} + \sum_{i=k+1}^{\infty} a_i \right\}\right).$$

Note that if  $\sigma_0^2$  and  $\eta_0^2$  have bounded densities, then  $f_{Y^2}(x) \leq C|\ln(x)|$  in a neighborhood of 0, so that Proposition 2.1(2) holds with  $\rho = 0$  and  $\alpha = 1$ .

Under the assumptions of Proposition 2.1(2), we obtain for Model (2.5) the following rates for  $((X_t, Z_t))_{t \geq 0}$ :

- If  $a_j = 0$ , for  $j \geq J$ , then  $((X_t, Z_t))_{t \geq 0}$  is geometrically  $\tau$ -dependent.
- If  $a_j = O(b^j)$  for some  $b < 1$  then  $\tau_\infty(n) = O(\kappa \sqrt{n})$  for some  $\kappa < 1$ .
- If  $a_j = O(j^{-b})$  for some  $b > 1$  then  $\tau_\infty(n) = O(n^{-b(1-\rho)/(2-\rho)} (\ln(n))^{(b+2)(1+\alpha)/2})$ .

For more general models than (2.5), we refer to Doukhan *et al.* (2006).

For Model (2.2), if there exists  $\kappa < 1$  such that

$$(2.18) \quad \mathbb{E}(|(f(x, \eta_0))^2 - (f(y, \eta_0))^2|) \leq \kappa|x^2 - y^2|,$$

then one can take  $\delta_n = 4\mathbb{E}(\sigma_0^2)\kappa^n$ . Hence, under the assumptions of Proposition 2.1(2),  $((X_t, Z_t))_{t \geq 0}$  is geometrically  $\tau$  dependent. An example of Markov chain satisfying (2.18) is the autoregressive model  $\sigma_t^2 = h(\sigma_{t-1}^2) + r(\eta_{t-1})$  for some  $\kappa$ -lipschitz function  $h$ .

### 3. THE ESTIMATORS

For two complex-valued functions  $u$  and  $v$  in  $\mathbb{L}_2(\mathbb{R}) \cap \mathbb{L}_1(\mathbb{R})$ , let  $u^*(x) = \int e^{itx} u(t) dt$ ,  $u * v(x) = \int u(y)v(x-y)dy$ , and  $\langle u, v \rangle = \int u(x)\overline{v(x)}dx$  with  $\overline{z}$  the conjugate of a complex number  $z$ . We also denote by  $\|u\|_1 = \int |u(x)|dx$ ,  $\|u\|^2 = \int |u(x)|^2 dx$ , and  $\|u\|_\infty = \sup_{x \in \mathbb{R}} |u(x)|$ .

**3.1. The projection spaces.** Let  $\varphi(x) = \sin(\pi x)/(\pi x)$ . For  $m \in \mathbb{N}$  and  $j \in \mathbb{Z}$ , set  $\varphi_{m,j}(x) = \sqrt{m}\varphi(mx - j)$ . The functions  $\{\varphi_{m,j}\}_{j \in \mathbb{Z}}$  constitute an orthonormal system in  $\mathbb{L}^2(\mathbb{R})$  (see e.g. Meyer (1990), p.22). Let us define

$$S_m = \overline{\text{span}}\{\varphi_{m,j}, j \in \mathbb{Z}\}, m \in \mathbb{N}.$$

The space  $S_m$  is exactly the subspace of  $\mathbb{L}_2(\mathbb{R})$  of functions having a Fourier transform with compact support contained in  $[-\pi m, \pi m]$ . The orthogonal projection of  $g$  on  $S_m$  is  $g_m = \sum_{j \in \mathbb{Z}} a_{m,j}(g) \varphi_{m,j}$  where  $a_{m,j}(g) = \langle \varphi_{m,j}, g \rangle$ . To obtain representations having a finite number of "coordinates", we introduce

$$S_m^{(n)} = \overline{\text{span}}\{\varphi_{m,j}, |j| \leq k_n\}$$

with integers  $k_n$  to be specified later. The family  $\{\varphi_{m,j}\}_{|j| \leq k_n}$  is an orthonormal basis of  $S_m^{(n)}$  and the orthogonal projections of  $g$  on  $S_m^{(n)}$  is given by  $g_m^{(n)} = \sum_{|j| \leq k_n} a_{m,j}(g) \varphi_{m,j}$ . Subsequently a space  $S_m^{(n)}$  will be referred to as a "model" as well as a "projection space".

**3.2. Construction of the minimum contrast estimators.** We subsequently assume that

$$(3.1) \quad f_\varepsilon \text{ belongs to } \mathbb{L}_2(\mathbb{R}) \text{ and is such that } \forall x \in \mathbb{R}, f_\varepsilon^*(x) \neq 0.$$

Note that the square integrability of  $f_\varepsilon$  and (1.3) require that  $\gamma > 1/2$  when  $\delta = 0$ . Under Condition (3.1) and for or  $t$  in  $S_m^{(n)}$ , we define the contrast function

$$\gamma_n(t) = \frac{1}{n} \sum_{i=1}^n [\|t\|^2 - 2u_t^*(Z_i)], \quad \text{with} \quad u_t(x) = \frac{1}{2\pi} \left( \frac{t^*(-x)}{f_\varepsilon^*(x)} \right).$$

Then, for an arbitrary fixed integer  $m$ , an estimator of  $g$  belonging to  $S_m^{(n)}$  is defined by

$$(3.2) \quad \hat{g}_m^{(n)} = \arg \min_{t \in S_m^{(n)}} \gamma_n(t).$$

By using Parseval and inverse Fourier formulae we obtain that  $\mathbb{E}[u_t^*(Z_i)] = \langle t, g \rangle$ , so that  $\mathbb{E}(\gamma_n(t)) = \|t - g\|^2 - \|g\|^2$  is minimal when  $t = g$ . This shows that  $\gamma_n(t)$  suits well for the estimation of  $g$ . It is easy to see that

$$\hat{g}_m^{(n)} = \sum_{|j| \leq k_n} \hat{a}_{m,j} \varphi_{m,j} \quad \text{with} \quad \hat{a}_{m,j} = \frac{1}{n} \sum_{i=1}^n u_{\varphi_{m,j}}^*(Z_i), \quad \text{and} \quad \mathbb{E}(\hat{a}_{m,j}) = \langle g, \varphi_{m,j} \rangle = a_{m,j}(g).$$

**3.3. Minimum penalized contrast estimator.** The minimum penalized estimator of  $g$  is defined as  $\tilde{g} = \hat{g}_{\hat{m}_g}^{(n)}$  where  $\hat{m}_g$  is chosen in a purely data-driven way. The main point of the estimation procedure lies in the choice of  $m = \hat{m}$  (or equivalently in the choice of model  $S_{\hat{m}}^{(n)}$ ) involved in the estimators  $\hat{g}_m^{(n)}$  given by (3.2), in order to mimic the oracle parameter

$$(3.3) \quad \check{m}_g = \arg \min_m \mathbb{E} \| \hat{g}_m^{(n)} - g \|^2_2.$$



The model selection is performed in an automatic way, using the following penalized criteria

$$(3.4) \quad \tilde{g} = \hat{g}_{\hat{m}}^{(n)} \text{ with } \hat{m} = \arg \min_{m \in \{1, \dots, m_n\}} \left[ \gamma_n(\hat{g}_m^{(n)}) + \text{pen}(m) \right],$$

where  $\text{pen}(m)$  is a penalty function that depends on  $f_\varepsilon^*(\cdot)$  through  $\Delta(m)$  defined by

$$(3.5) \quad \Delta(m) = \frac{1}{2\pi} \int_{-\pi m}^{\pi m} \frac{1}{|f_\varepsilon^*(x)|^2} dx.$$

The key point in the dependent context is to find a penalty function not depending on the dependency coefficients such that

$$\mathbb{E} \|\tilde{g} - g\|^2 \leq C \inf_{m \in \{1, \dots, m_n\}} \mathbb{E} \|\hat{g}_m^{(n)} - g\|^2.$$

In that way, the estimator  $\tilde{g}$  is adaptive since it achieves the best rate among the estimators  $\hat{g}_m^{(n)}$ , without any prior knowledge on the smoothness on  $g$ .

#### 4. DENSITY ESTIMATION BOUNDS

From now on, the dependence coefficients are defined as in (2.9), (2.10) and (2.11) with  $(W_t)_{t \in \mathbb{Z}} = ((Z_t, X_t))_{t \in \mathbb{Z}}$ .

**4.1. Rates of convergence of the minimum contrast estimators  $\hat{g}_m^{(n)}$ .** Subsequently, the density  $g$  is assumed to satisfy the following assumption:

$$(4.1) \quad g \in \mathbb{L}_2(\mathbb{R}), \text{ and there exists } M_2 > 0, \int x^2 g^2(x) dx \leq M_2 < \infty.$$

Assumption (4.1), which is due to the construction of the estimator, already appears in density deconvolution in the independent framework in Comte *et al.* (2005, 2006). It is important to note that Assumption (4.1) is very unrestrictive. In particular, all densities having tails of order  $|x|^{-(s+1)}$  as  $x$  tends to infinity satisfy (4.1) only if  $s > 1/2$ . One can cite for instance the Cauchy distribution or all stable distributions with exponent  $r > 1/2$  (see Devroye (1986)). The Lévy distribution, with exponent  $r = 1/2$  does not satisfies (4.1).

Note that (4.1) is fulfilled if  $g$  is bounded by  $M_0$  and  $\mathbb{E}(X_1^2) \leq M_1 < +\infty$ , with  $M_2 = M_0 M_1$ .

The order of the MISE of  $\hat{g}_m^{(n)}$  is given in the following proposition.

**Proposition 4.1.** *If (3.1) and (4.1) hold, then  $\hat{g}_m^{(n)}$  defined by (3.2) satisfies*

$$\mathbb{E} \|g - \hat{g}_m^{(n)}\|^2 \leq \|g - g_m\|^2 + \frac{m^2(M_2 + 1)}{k_n} + \frac{2\Delta(m)}{n} + \frac{2R_m}{n},$$

where

$$(4.2) \quad R_m = \frac{1}{\pi} \sum_{k=2}^n \int_{-\pi m}^{\pi m} \left| \frac{\text{Cov}(e^{ixZ_1}, e^{ixX_k})}{f_\varepsilon^*(-x)} \right| dx.$$

Moreover,  $R_m \leq \min(R_{m,\beta}, R_{m,\tau})$ , where

$$R_{m,\beta} = 4\Delta_{1/2}(m) \sum_{k=1}^{n-1} \beta_1(k) \quad \text{and} \quad R_{m,\tau} = \pi m \Delta_{1/2}(m) \sum_{k=1}^{n-1} \tau_1(k),$$

with  $\beta_1, \tau_1$  defined by (2.9), and where

$$(4.3) \quad \Delta_{1/2}(m) = \frac{1}{2\pi} \int_{-\pi m}^{\pi m} \frac{1}{|f_\varepsilon^*(x)|} dx.$$

This proposition requires several comments.

As usual, the order of the risk is given by a bias term  $\|g_m - g\|^2 + m^2(M_2 + 1)/k_n$  and a variance term  $2\Delta(m)/n + 2R_m/n$ . As in density deconvolution for i.i.d. variables, the variance term  $2\Delta(m)/n + 2R_m/n$  depends on the rate of decay of the Fourier transform of  $f_\varepsilon$ . It is the sum of the variance term appearing in density deconvolution for i.i.d. variables  $2\Delta(m)/n$  and of an additional term  $2R_m/n$ . This last term  $R_m$  involves the dependency coefficients and the quantity  $\Delta_{1/2}(m)$ , which is specific to the ARCH problem. The point is that, as in the i.i.d. case, the main order term in the variance part is  $\Delta(m)/n$ , which does not involve the dependency coefficients. In other words, the dependency coefficients only appear in front of the additional and negligible term  $\Delta_{1/2}(m)/n$ , specific to ARCH models.

The bias term is the sum of the usual bias term  $\|g_m - g\|^2$ , depending on the smoothness properties of  $g$ , and on an additional term  $m^2(M_2 + 1)/k_n$ . With a suitable choice of  $k_n$ , not depending on  $g$ , this last term is negligible with respect to the variance term.

Concerning the main variance term,  $\Delta(m)$  given by (3.5) has the same order as

$$\Gamma(m) = (1 + (\pi m)^2)^\gamma (\pi m)^{1-\delta} \exp\{2\mu(\pi m)^\delta\},$$

up to some constant bounded by

$$(4.4) \quad \lambda_1(f_\varepsilon, \kappa_0) = \frac{1}{\kappa_0^2 \pi R(\mu, \delta)}, \quad \text{where } R(\mu, \delta) = \mathbb{I}_{\{\delta=0\}} + 2\mu\delta\mathbb{I}_{\{\delta>0\}}.$$

The rates resulting from Proposition 4.1 under (1.3) and (1.4) are given in the following proposition.

**Corollary 4.1.** *Assume that (1.3), (3.1), and (4.1) hold, that  $g$  belongs to  $\mathcal{S}_{s,r,b}(C_1)$  defined by (1.4), and that  $k_n \geq n$ . Assume either that*

- (1)  $\sum_{k \geq 1} \beta_1(k) < +\infty$
- (2) or  $\delta = 0$ ,  $\gamma > 1$  in (1.3) and  $\sum_{k \geq 1} \tau_1(k) < +\infty$
- (3) or  $\delta > 0$  in (1.3) and  $\sum_{k \geq 1} \tau_1(k) < +\infty$ .

Then  $\hat{g}_m^{(n)}$  defined by (3.2) satisfies

$$(4.5) \quad \mathbb{E}\|g - \hat{g}_m^{(n)}\|^2 \leq \frac{C_1}{2\pi} (m^2 \pi^2 + 1)^{-s} \exp\{-2b\pi^r m^r\} + \frac{2\lambda_1(f_\varepsilon, \kappa_0)\Gamma(m)}{n} + \frac{C_2}{n} \Gamma(m) o_m(1),$$

where  $C_1$  and  $C_2$  are finite constants. The constant  $C_2$  depends on  $\sum_{k \geq 1} \beta_1(k)$  (respectively on  $\sum_{k \geq 1} \tau_1(k)$ ).

If  $\gamma = 1$  when  $\delta = 0$ , then the bound 4.5 becomes

$$(4.6) \quad \mathbb{E}\|g - \hat{g}_m^{(n)}\|^2 \leq \frac{C_1}{2\pi} (m^2 \pi^2 + 1)^{-s} \exp\{-2b\pi^r m^r\} + \frac{(2 + C_2)\lambda_1(f_\varepsilon, \kappa_0)\Gamma(m)}{n},$$

with  $C_2$  depending on  $\sum_{k \geq 1} \beta_1(k)$  (respectively on  $\sum_{k \geq 1} \tau_1(k)$ ).

The rate of convergence of  $\hat{g}_m^{(n)}$  is the same as the rate for density deconvolution for i.i.d. sequences. Our context here encompasses the particular case considered by van Es *et al.* (2005).

Table 1 below gives a summary of these rates obtained when minimizing the right hand of (4.5). The  $\check{m}_g$  denotes the corresponding minimizer (see 3.3).

TABLE 1. Choice of  $\check{m}_g$  and corresponding rates under Assumptions (1.3) and (1.4).

		$f_\varepsilon$	
		$\delta = 0$ ordinary smooth	$\delta > 0$ supersmooth
	$r = 0$	$\pi\check{m}_g = O(n^{1/(2s+2\gamma+1)})$	$\pi\check{m}_g = [\ln(n)/(2\mu+1)]^{1/\delta}$
	Sobolev(s)	rate = $O(n^{-2s/(2s+2\gamma+1)})$	rate = $O((\ln(n))^{-2s/\delta})$
$g$	$r > 0$	$\pi\check{m}_g = [\ln(n)/2b]^{1/r}$	$\check{m}_g$ solution of $\check{m}_g^{2s+2\gamma+1-r} \exp\{2\mu(\pi\check{m}_g)^\delta + 2b\pi^r \check{m}_g^r\}$ $= O(n)$
	$\mathcal{C}^\infty$	rate = $O\left(\frac{\ln(n)^{(2\gamma+1)/r}}{n}\right)$	

When  $r > 0, \delta > 0$  the value of  $\check{m}_g$  is not explicitly given. It is obtained as the solution of the equation

$$\check{m}_g^{2s+2\gamma+1-r} \exp\{2\mu(\pi\check{m}_g)^\delta + 2b\pi^r \check{m}_g^r\} = O(n).$$

Consequently, the rate of  $\hat{g}_{\check{m}_g}^{(n)}$  is not easy to give explicitly and depends on the ratio  $r/\delta$ . If  $r/\delta$  or  $\delta/r$  belongs to  $]k/(k+1); (k+1)/(k+2)[$  with  $k$  integer, the rate of convergence can be expressed as a function of  $k$ . We refer to Comte *et al.* (2006) for further discussions about those rates. We refer to Lacour (2006) for explicit formulae for the rates in the special case  $r > 0$  and  $\delta > 0$ .

**4.2. Adaptive bound.** Theorem 4.1 below gives a general bound which holds under weak dependency conditions, for  $\varepsilon$  being either ordinary or super smooth.

For  $a > 1$ , let  $\text{pen}(m)$  be defined by

$$(4.7) \quad \text{pen}(m) = \begin{cases} 192a \frac{\Delta(m)}{n} & \text{if } 0 \leq \delta < 1/3, \\ 64a\lambda_3 \frac{\Delta(m) m^{\min((3\delta/2-1/2)+, \delta)}}{n} & \text{if } \delta \geq 1/3, \end{cases}$$

where  $\Delta(m)$  is defined by (3.5). The constant  $\lambda_1(f_\varepsilon, \kappa_0)$  is defined in (4.4) and

$$(4.8) \quad \lambda_3 = 1 + \frac{32\mu\pi^\delta}{\lambda_1(f_\varepsilon, \kappa_0')} \left( (\sqrt{2} + 8) \|f_\varepsilon\|_\infty \kappa_0^{-1} \sqrt{\lambda_1(f_\varepsilon, \kappa_0)} \mathbb{I}_{0 \leq \delta \leq 1} + 2\lambda_1(f_\varepsilon, \kappa_0) \mathbb{I}_{\delta > 1} \right).$$

The important point here is that  $\lambda_3$  is known. Hence the penalty is explicit up to a numerical multiplicative constant. This procedure has already been practically studied for independent sequences  $(X_t)_{t \geq 1}$  and  $(\varepsilon_t)_{t \geq 1}$  in Comte *et al.* (2005, 2006). In particular, the practical implementation of the penalty functions, and the calibration of the constants have been studied in the two previously mentioned papers. Moreover, it is shown therein that the estimation procedure is robust to various types of dependence, whether the errors  $\varepsilon_i$ 's are ordinary or super smooth (see Tables 4 and 5 in Comte *et al.* (2005)).

In order to bound up  $\text{pen}(m)$ , we impose that

$$(4.9) \quad \pi m_n \leq \begin{cases} n^{1/(2\gamma+1)} & \text{if } \delta = 0 \\ \left[ \frac{\ln(n)}{2\mu} + \frac{2\gamma+1-\delta}{2\delta\mu} \ln \left( \frac{\ln(n)}{2\mu} \right) \right]^{1/\delta} & \text{if } \delta > 0. \end{cases}$$

Subsequently we set

$$(4.10) \quad C_a = \max(\kappa_a^2, 2\kappa_a) \text{ where } \kappa_a = (a+1)/(a-1).$$

**Theorem 4.1.** *Assume that  $f_\varepsilon$  satisfies (1.3) and 3.1, that  $g$  satisfies (4.1), and that  $m_n$  satisfies (4.9). Let  $\text{pen}(m)$  be defined by (4.7). Consider the collection of estimators  $\hat{g}_m^{(n)}$  defined by (3.2) with  $k_n \geq n$  and  $1 \leq m \leq m_n$ . Let  $\beta_\infty$  and  $\tau_\infty$  be defined as in (2.10) and (2.11) respectively. Assume either that*

- (1)  $\beta_\infty(k) = O(k^{-(1+\theta)})$  for some  $\theta > 3$
- (2) or  $\delta = 0$ ,  $\gamma \geq 3/2$  in (1.3) and  $\tau_\infty(k) = O(k^{-(1+\theta)})$  for some  $\theta > 3 + 2/(1+2\gamma)$
- (3) or  $\delta > 0$  in (1.3) and  $\tau_\infty(k) = O(k^{-(1+\theta)})$  for some  $\theta > 3$ .

Then the estimator  $\tilde{g} = \hat{g}_m^{(n)}$  defined by (3.4) satisfies

$$(4.11) \quad \mathbb{E}(\|g - \tilde{g}\|^2) \leq C_a \inf_{m \in \{1, \dots, m_n\}} \left[ \|g - g_m\|^2 + \text{pen}(m) + \frac{m^2(M_2 + 1)}{n} \right] + \frac{\overline{C}}{n},$$

where  $C_a$  is defined in (4.10) and  $\overline{C}$  is a constant depending on  $f_\varepsilon$ ,  $a$ , and  $\sum_{k \geq 1} \beta_\infty(k)$  (respectively on  $\sum_{k \geq 1} \tau_\infty(k)$ ).

**Remark 4.1.** In case (2), when  $\delta = 0$  in (1.3), the condition on  $\theta$  is weaker as  $\gamma$  increases and  $f_\varepsilon$  gets smoother.

The estimator  $\tilde{g}$  is adaptive in the sense that it is purely data-driven. This is due to the fact that  $\text{pen}(\cdot)$  is explicitly known. In particular, its construction does not require any prior smoothness knowledge on the unknown density  $g$  and does not use the dependency coefficients. This point is important since all quantities involving dependency coefficients are usually not tractable in practice.

The main result in Theorem 4.1 shows that the MISE of  $\tilde{g}$  automatically achieves the best squared-bias variance compromise (possibly up to some logarithmic factor). Consequently, it achieves the best rate among the rates of the  $\hat{g}_m^{(n)}$ , even from a non-asymptotical point of view. This last point is of most importance since the  $m$  selected in practice are small and far away from asymptotic. For practical illustration of this point in the case of density deconvolution of i.i.d. variables, we refer to Comte *et al.* (2005, 2006). Another important point is that, if we consider the asymptotic trade-off, then the rates given in

Table 1 are automatically reached in most cases by the adaptive estimator  $\tilde{g}$ . Only in the case  $\delta > 1/3$  and  $r > 0$ , a loss may occur in the rate of  $\tilde{g}$ . This comes from the additional power of  $m$  in the penalty for  $\delta \geq 1/3$  with respect to the variance order  $\Delta(m)$ . Nevertheless, the resulting loss in the rate has an order which is negligible compared to the main order rate.

As a conclusion, the estimator  $\tilde{g}$  has the rate of the i.i.d. case, with an explicit penalty function not depending on the dependency coefficients.

## 5. PROOFS

**5.1. Proof of Proposition 4.1.** The proof of Proposition 4.1 follows the same lines as in the independent framework (see Comte *et al.* (2006)). The main difference lies in the control of the variance term. We keep the same notations as in Section 3.2. According to (3.2), for any given  $m$  belonging to  $\{1, \dots, m_n\}$ ,  $\hat{g}_m^{(n)}$  satisfies,  $\gamma_n(\hat{g}_m^{(n)}) - \gamma_n(g_m^{(n)}) \leq 0$ . For a random variable  $T$  with density  $f_T$ , and any function  $\psi$  such that  $\psi(T)$  is integrable, set  $\nu_{n,T}(\psi) = n^{-1} \sum_{i=1}^n [\psi(T_i) - \langle \psi, f_T \rangle]$ . In particular,

$$(5.1) \quad \nu_{n,Z}(u_t^*) = \frac{1}{n} \sum_{i=1}^n [u_t^*(Z_i) - \langle t, g \rangle].$$

Since

$$(5.2) \quad \gamma_n(t) - \gamma_n(s) = \|t - g\|^2 - \|s - g\|^2 - 2\nu_{n,Z}(u_{t-s}^*),$$

we infer that

$$(5.3) \quad \|g - \hat{g}_m^{(n)}\|^2 \leq \|g - g_m^{(n)}\|^2 + 2\nu_{n,Z} \left( u_{\hat{g}_m^{(n)} - g_m^{(n)}}^* \right).$$

Writing that  $\hat{a}_{m,j} - a_{m,j} = \nu_{n,Z}(u_{\varphi_{m,j}}^*)$ , we obtain that

$$\nu_{n,Z} \left( u_{\hat{g}_m^{(n)} - g_m^{(n)}}^* \right) = \sum_{|j| \leq k_n} (\hat{a}_{m,j} - a_{m,j}) \nu_{n,Z}(u_{\varphi_{m,j}}^*) = \sum_{|j| \leq k_n} [\nu_{n,Z}(u_{\varphi_{m,j}}^*)]^2.$$

Consequently,  $\mathbb{E}\|g - \hat{g}_m^{(n)}\|^2 \leq \|g - g_m^{(n)}\|^2 + 2 \sum_{j \in \mathbb{Z}} \mathbb{E}[(\nu_{n,Z}(u_{\varphi_{m,j}}^*))^2]$ . According to Comte *et al.* (2006),

$$(5.4) \quad \|g - g_m^{(n)}\|^2 = \|g - g_m\|^2 + \|g_m - g_m^{(n)}\|^2 \leq \|g - g_m\|^2 + \frac{(\pi m)^2 (M_2 + 1)}{k_n}.$$

The variance term is studied by using first that for  $f \in \mathbb{L}_1(\mathbb{R})$ ,

$$(5.5) \quad \nu_{n,Z}(f^*) = \int \nu_{n,Z}(e^{ix \cdot}) f(x) dx.$$

Now, we use (5.5) and apply Parseval's formula to obtain

$$(5.6) \quad \begin{aligned} \mathbb{E} \left( \sum_{j \in \mathbb{Z}} (\nu_{n,Z}(u_{\varphi_{m,j}}^*))^2 \right) &= \frac{1}{4\pi^2} \sum_{j \in \mathbb{Z}} \mathbb{E} \left( \int \frac{\varphi_{m,j}^*(-x)}{f_\varepsilon^*(x)} \nu_{n,Z}(e^{ix \cdot}) dx \right)^2 \\ &= \frac{1}{2\pi} \int_{-\pi m}^{\pi m} \frac{\mathbb{E} |\nu_{n,Z}(e^{ix \cdot})|^2}{|f_\varepsilon^*(x)|^2} dx. \end{aligned}$$

Since  $\nu_{n,Z}$  involves centered and stationary variables, we have

$$(5.7) \quad \mathbb{E}|\nu_{n,Z}(e^{ix\cdot})|^2 = \text{Var}|\nu_{n,Z}(e^{ix\cdot})| = \frac{1}{n} \text{Var}(e^{ixZ_1}) + \frac{1}{n^2} \sum_{1 \leq k \neq l \leq n} \text{Cov}(e^{ixZ_k}, e^{ixZ_l}).$$

It follows from the structure of the model that, for  $k < l$ ,  $\varepsilon_l$  is independent of  $(X_l, Z_k)$ , so that  $\mathbb{E}(e^{ixZ_k}) = f_\varepsilon^*(x)g^*(x)$  and  $\mathbb{E}(e^{ix(Z_l - Z_k)}) = f_\varepsilon^*(x)\mathbb{E}(e^{ix(X_l - Z_k)})$ . Thus, for  $k < l$ ,

$$(5.8) \quad \text{Cov}(e^{ixZ_k}, e^{ixZ_l}) = f_\varepsilon^*(x) \text{Cov}(e^{ixZ_k}, e^{ixX_l}).$$

From (5.7) and the stationarity of  $(X_i)_{i \geq 1}$ , we obtain that

$$(5.9) \quad \mathbb{E}|\nu_{n,Z}(e^{ix\cdot})|^2 \leq \frac{1}{n} + \frac{2}{n} \sum_{k=2}^n |\text{Cov}(e^{ixZ_1}, e^{ixX_k})| |f_\varepsilon^*(x)|.$$

The first part of Proposition 4.1 follows from the stationarity of the  $X_i$ 's, and from (5.3), (5.4), (5.6) and (5.9).

The proof of  $R_m \leq \min(R_{m,\beta}, R_{m,\tau})$ , where  $R_{m,\beta}$  and  $R_{m,\tau}$  are defined in Proposition 4.1, comes from the inequalities (2.15) in Section 2.2. Hence we get the result.  $\square$

**5.2. Proof of Corollary 4.1.** According to Butucea and Tsybakov (2005), under (1.3), we have

$$\lambda_1(f_\varepsilon, \kappa'_0)\Gamma(m)(1 + o_m(1)) \leq \Delta(m) \leq \lambda_1(f_\varepsilon, \kappa_0)\Gamma(m)(1 + o_m(1)) \quad \text{as } m \rightarrow \infty, \text{ where}$$

$$(5.10) \quad \Gamma(m) = (1 + (\pi m)^2)^\gamma (\pi m)^{1-\delta} \exp\{2\mu(\pi m)^\delta\},$$

where  $\lambda_1$  is defined in (4.4). In the same way

$$\overline{\lambda}_1(f_\varepsilon, \kappa'_0)\overline{\Gamma}(m)(1 + o_m(1)) \leq \Delta_{1/2}(m) \leq \overline{\lambda}_1(f_\varepsilon, \kappa_0)\overline{\Gamma}(m)(1 + o_m(1)) \quad \text{as } m \rightarrow \infty,$$

where

$$\begin{aligned} \overline{\Gamma}(m) &= (1 + (\pi m)^2)^{\gamma/2} (\pi m)^{1-\delta} \exp(\mu(\pi m)^\delta) \\ \overline{\lambda}_1(f_\varepsilon, \kappa_0) &= [\kappa_0^2 \pi (\mathbb{I}_{\{\delta=0\}} + \mu \delta \mathbb{I}_{\{\delta>0\}})]^{-1}. \end{aligned}$$

It is easy to see that  $\Delta_{1/2}(m) \leq \sqrt{m\Delta(m)}$  and hence  $\Delta_{1/2}(m) = \Gamma(m)o_m(1)$ . Now, as soon as  $\gamma > 1$  when  $\delta = 0$ ,  $m\Delta_{1/2}(m) = \Gamma(m)o_m(1)$ . Set  $m_1$  such that for  $m \geq m_1$  we have

$$(5.11) \quad 0.5\lambda_1(f_\varepsilon, \kappa'_0)\Gamma(m) \leq \Delta(m) \leq 2\lambda_1(f_\varepsilon, \kappa_0)\Gamma(m),$$

and

$$(5.12) \quad 0.5\overline{\lambda}_1(f_\varepsilon, \kappa'_0)\overline{\Gamma}(m) \leq \Delta_{1/2}(m) \leq 2\overline{\lambda}_1(f_\varepsilon, \kappa_0)\overline{\Gamma}(m).$$

If  $\sum_{k \geq 1} \beta_1(k) < +\infty$ , (1.3) and (4.1) hold, and if  $k_n \geq n$ , then we have the upper bounds: for  $m \geq m_1$ ,  $\lambda_1 = \lambda_1(f_\varepsilon, \kappa_0)$  and  $\overline{\lambda}_1 = \overline{\lambda}_1(f_\varepsilon, \kappa_0)$ ,

$$\begin{aligned} \mathbb{E}\|g - \hat{g}_m^{(n)}\|^2 &\leq \|g - g_m\|^2 + \frac{m^2(M_2 + 1)}{n} + \frac{2\lambda_1\Gamma(m)}{n} + 8\overline{\lambda}_1 \sum_{k \geq 1} \beta_1(k) \frac{\overline{\Gamma}(m)}{n} \\ &\leq \|g - g_m\|^2 + \frac{m^2(M_2 + 1)}{n} + \frac{2\lambda_1\Gamma(m)}{n} + \frac{C(\sum_{k \geq 1} \beta_1(k))\Gamma(m)}{n} o_m(1). \end{aligned}$$

In the same way, if  $\sum_{k \geq 1} \tau_1(k) < +\infty$ , if  $\gamma > 1$  when  $\delta = 0$ , if (1.3) and (4.1) hold, and if  $k_n \geq n$ , then we have the upper bound: for  $m \geq m_1$ ,

$$\begin{aligned} \mathbb{E}\|g - \hat{g}_m^{(n)}\|^2 &\leq \|g - g_m\|^2 + \frac{m^2(M_2 + 1)}{n} + \frac{2\lambda_1\Gamma(m)}{n} + 2\pi\overline{\lambda_1} \sum_{k \geq 1} \tau_1(k) \frac{m\overline{\Gamma}(m)}{n} \\ &\leq \|g - g_m\|^2 + \frac{m^2(M_2 + 1)}{n} + \frac{2\lambda_1\Gamma(m)}{n} + \frac{C(\sum_{k \geq 1} \tau_1(k))\Gamma(m)}{n} o_m(1). \end{aligned}$$

Since  $\gamma > 1$  when  $\delta = 0$ , the residual term  $n^{-1}m^2(M_2 + 1)$  is negligible with respect to the variance term.

Finally,  $g_m$  being the orthogonal projection of  $g$  on  $S_m$ , we get  $g_m^* = g^* \Pi_{[-m\pi, m\pi]}$  and therefore

$$\|g - g_m\|^2 = \frac{1}{2\pi} \|g^* - g_m^*\|^2 = \frac{1}{2\pi} \int_{|x| \geq \pi m} |g^*|^2(x) dx.$$

If  $g$  belongs to the class  $\mathcal{S}_{s,r,b}(C_1)$  defined in (1.4), then

$$\|g - g_m\|^2 \leq \frac{C_1}{2\pi} (m^2\pi^2 + 1)^{-s} \exp\{-2b\pi^r m^r\}.$$

The corollary is proved.  $\square$

**5.3. Proof of Theorem 4.1.** By definition,  $\tilde{g}$  satisfies that for all  $m \in \{1, \dots, m_n\}$ ,

$$\gamma_n(\tilde{g}) + \text{pen}(\hat{m}) \leq \gamma_n(g_m) + \text{pen}(m).$$

Therefore, by using (5.2) we get

$$\|\tilde{g} - g\|^2 \leq \|g_m^{(n)} - g\|^2 + 2\nu_{n,Z}(u_{\tilde{g}-g_m^{(n)}}^*) + \text{pen}(m) - \text{pen}(\hat{m}),$$

where  $\nu_{n,Z}$  is defined in (5.1). If  $t = t_1 + t_2$  with  $t_1$  in  $S_m^{(n)}$  and  $t_2$  in  $S_{m'}^{(n)}$ ,  $t^*$  has its support in  $[-\pi \max(m, m'), \pi \max(m, m')]$  and  $t$  belongs to  $S_{\max(m, m')}^{(n)}$ . Set  $B_{m, m'}(0, 1) = \{t \in S_{\max(m, m')}^{(n)} / \|t\| = 1\}$  and write

$$|\nu_{n,Z}(u_{\tilde{g}-g_m^{(n)}}^*)| \leq \|\tilde{g} - g_m^{(n)}\| \sup_{t \in B_{m, \hat{m}}(0, 1)} |\nu_{n,Z}(u_t^*)|.$$

Using that  $2uv \leq a^{-1}u^2 + av^2$  for any  $a > 1$ , leads to

$$\|\tilde{g} - g\|^2 \leq \|g_m^{(n)} - g\|^2 + a^{-1}\|\tilde{g} - g_m^{(n)}\|^2 + a \sup_{t \in B_{m, \hat{m}}(0, 1)} (\nu_{n,Z}(u_t^*))^2 + \text{pen}(m) - \text{pen}(\hat{m}).$$

**Proof in the  $\beta$ -mixing case.**

We use the coupling methods recalled in Section 2.2 to build approximating variables for the  $W_i = (Z_i, X_i)$ 's. More precisely, we build variables  $W_i^*$  such that if  $n = 2p_n q_n + r_n$ ,  $0 \leq r_n < q_n$ , and  $\ell = 0, \dots, p_n - 1$

$$\begin{aligned} E_\ell &= (W_{2\ell q_n + 1}, \dots, W_{(2\ell+1)q_n}), \quad F_\ell = (W_{(2\ell+1)q_n + 1}, \dots, W_{(2\ell+2)q_n}), \\ E_\ell^* &= (W_{2\ell q_n + 1}^*, \dots, W_{(2\ell+1)q_n}^*), \quad F_\ell^* = (W_{(2\ell+1)q_n + 1}^*, \dots, W_{(2\ell+2)q_n}^*). \end{aligned}$$

The variables  $E_\ell^*$  and  $F_\ell^*$  are such that

- $E_\ell^*$  and  $E_\ell$  are identically distributed.  $F_\ell^*$  and  $F_\ell$  are identically distributed.
- $\mathbb{P}(E_\ell \neq E_\ell^*) \leq \beta_\infty(q_n)$  and  $\mathbb{P}(F_\ell \neq F_\ell^*) \leq \beta_\infty(q_n)$ ,

-  $E_\ell^*$  and  $\mathcal{M}_0 \vee \sigma(E_0, E_1, \dots, E_{\ell-1}, E_0^*, E_1^*, \dots, E_{\ell-1}^*)$  are independent, and therefore independent of  $\mathcal{M}_{(\ell-1)q_n}$  and the same holds for the blocks  $F_\ell^*$ .

For the sake of simplicity we assume that  $r_n = 0$ . We denote by  $(Z_i^*, X_i^*) = W_i^*$  the new couple of variables. We start from

$$(5.13) \quad \|\tilde{g} - g\|^2 \leq \kappa_a^2 \|g_m^{(n)} - g\|^2 + a\kappa_a \sup_{t \in B_{m, \hat{m}}(0,1)} |\nu_{n,Z}(u_t^*)|^2 + \kappa_a(\text{pen}(m) - \text{pen}(\hat{m})),$$

where  $\kappa_a$  is defined in (4.10). Using the notation (5.1), we denote by  $\nu_{n,Z}^*(u_t^*)$  the empirical contrast computed on the  $Z_i^*$ . Then we write

$$\begin{aligned} \|\tilde{g} - g\|^2 &\leq \kappa_a^2 \|g - g_m^{(n)}\|^2 + 2a\kappa_a \sup_{t \in B_{m, \hat{m}}(0,1)} |\nu_{n,Z}^*(u_t^*)|^2 + \kappa_a(\text{pen}(m) - \text{pen}(\hat{m})) \\ &\quad + 2a\kappa_a \sup_{t \in B_{m, \hat{m}}(0,1)} |\nu_{n,Z}^*(u_t^*) - \nu_{n,Z}(u_t^*)|^2. \end{aligned}$$

Set

$$(5.14) \quad T_n^*(m, m') := \left[ \sup_{t \in B_{m, m'}(0,1)} |\nu_{n,Z}^*(t)|^2 - p(m, m') \right]_+.$$

Hence

$$\begin{aligned} \|\tilde{g} - g\|^2 &\leq \kappa_a^2 \|g - g_m^{(n)}\|^2 + 2a\kappa_a T_n^*(m, \hat{m}) + \kappa_a(2ap(m, \hat{m}) + \text{pen}(m) - \text{pen}(\hat{m})) \\ &\quad + 2a\kappa_a \sup_{t \in B_{m, \hat{m}}(0,1)} |\nu_{n,Z}(u_t^*) - \nu_{n,Z}^*(u_t^*)|^2 \\ &\leq \kappa_a^2 \|g - g_m^{(n)}\|^2 + 2\kappa_a \text{pen}(m) + 2a\kappa_a \sup_{t \in B_{m, \hat{m}}(0,1)} |\nu_{n,Z}(u_t^*) - \nu_{n,Z}^*(u_t^*)|^2 \\ (5.15) \quad &\quad + 2a\kappa_a T_n^*(m, \hat{m}) \end{aligned}$$

where  $\text{pen}(m)$  is chosen such that

$$(5.16) \quad 2ap(m, m') \leq \text{pen}(m) + \text{pen}(m').$$

Now write

$$\begin{aligned} \nu_{n,Z}(u_t^*) - \nu_{n,Z}^*(u_t^*) &= \frac{1}{2\pi} \frac{1}{n} \sum_{k=1}^n \int [e^{ixZ_k} - e^{ixZ_k^*}] \frac{t^*(-x)}{f_\varepsilon^*(x)} dx \\ &= \frac{1}{2\pi} \int [\nu_{n,Z}(e^{ix\cdot}) - \nu_{n,Z}^*(e^{ix\cdot})] \frac{t^*(-x)}{f_\varepsilon^*(x)} dx. \end{aligned}$$

Consequently,

$$(5.17) \quad \mathbb{E} \left[ \sup_{t \in B_{m, \hat{m}}(0,1)} |\nu_{n,Z}(u_t^*) - \nu_{n,Z}^*(u_t^*)|^2 \right] \leq \int_{-\pi m_n}^{\pi m_n} \mathbb{E}[|\nu_{n,Z}(e^{ix\cdot}) - \nu_{n,Z}^*(e^{ix\cdot})|^2] \frac{1}{|f_\varepsilon^*(x)|^2} dx.$$

Since

$$\begin{aligned} \mathbb{E}[|\nu_{n,Z}(e^{ix\cdot}) - \nu_{n,Z}^*(e^{ix\cdot})|^2] &= \mathbb{E}[|\nu_{n,Z}(e^{ix\cdot}) - \nu_{n,Z}^*(e^{ix\cdot}) \mathbb{1}_{Z_k \neq Z_k^*}|^2] \\ &\leq 4\mathbb{E} \left[ \frac{1}{n} \sum_{k=1}^n |\mathbb{1}_{Z_k \neq Z_k^*}|^2 \right] \leq 4\beta_\infty(q_n), \end{aligned}$$



we obtain that

$$(5.18) \quad \mathbb{E} \left[ \sup_{t \in B_{m, \hat{m}}(0,1)} |\nu_{n,Z}(u_t^*) - \nu_{n,Z}^*(u_t^*)|^2 \right] \leq 4\beta_\infty(q_n)\Delta(m_n).$$

By gathering (5.15) and (5.18) we get

$$\mathbb{E} \|\tilde{g} - g\|^2 \leq \kappa_a^2 \|g - g_m^{(n)}\|^2 + 2a\kappa_a \sum_{m'=1}^{m_n} \mathbb{E}[T_n^*(m, m')] + 2\kappa_a \text{pen}(m) + 2a\kappa_a \beta_\infty(q_n)\Delta(m_n).$$

Therefore we infer that, for all  $m \in \{1, \dots, m_n\}$ ,

$$(5.19) \quad \mathbb{E} \|g - \tilde{g}\|^2 \leq C_a \left[ \|g - g_m^{(n)}\|^2 + \text{pen}(m) \right] + 2a\kappa_a(C_1 + C_2)/n,$$

provided that

$$(5.20) \quad \Delta(m_n)\beta_\infty(q_n) \leq C_1/n \quad \text{and} \quad \sum_{m'=1}^{m_n} \mathbb{E}(T_n^*(m, m')) \leq C_2/n.$$

Using (5.11), we conclude that the first part of (5.20) is fulfilled as soon as

$$(5.21) \quad m_n^{2\gamma+1-\delta} \exp\{2\mu\pi^\delta m_n^\delta\} \beta_\infty(q_n) \leq C'_1/n.$$

In order to ensure that our estimators converge, we only consider models with bounded penalty, and therefore (5.21) requires that  $\beta_\infty(q_n) \leq C'_1/n^2$ . For  $q_n = [n^c]$  and  $\beta_\infty(k) = O(n^{-1-\theta})$ , we obtain the condition  $n^{-c(1+\theta)} = O(n^{-2})$ . If  $\theta > 3$ , one can find  $c \in ]0, 1/2[$ , such that this condition is satisfied. Consequently, (5.21) holds.

To prove the second part of (5.20), we split  $T_n^*(m, m')$  into two terms

$$T_n^*(m, m') = (T_{n,1}^*(m, m') + T_{n,2}^*(m, m'))/2,$$

where, for  $k = 1, 2$ ,

$$(5.22) \quad T_{n,k}^*(m, m') = \left[ \sup_{t \in B_{m, m'}(0,1)} \left| \frac{1}{p_n q_n} \sum_{\ell=1}^{p_n} \sum_{i=1}^{q_n} (u_t^*(Z_{(2\ell+k-1)q_n+i}^*) - \langle t, g \rangle) \right|^2 - p_k(m, m') \right]_+.$$

We only study  $T_{n,1}^*(m, m')$  and conclude for  $T_{n,2}^*(m, m')$  analogously. The study of  $T_{n,1}^*(m, m')$  consists in applying a concentration inequality to  $\nu_{n,1}^*(t)$  defined by

$$(5.23) \quad \nu_{n,1}^*(t) = \frac{1}{p_n q_n} \sum_{\ell=1}^{p_n} \sum_{i=1}^{q_n} (u_t^*(Z_{2\ell q_n+i}^*) - \langle t, g \rangle) = \frac{1}{p_n} \sum_{\ell=1}^{p_n} \nu_{q_n, \ell}^*(u_t^*).$$

The random variable  $\nu_{n,1}^*(u_t^*)$  is considered as the sum of the  $p_n$  independent random variables  $\nu_{q_n, \ell}^*(t)$  defined as

$$(5.24) \quad \nu_{q_n, \ell}^*(u_t^*) = (1/q_n) \sum_{j=1}^{q_n} u_t^*(Z_{2\ell q_n+j}^*) - \langle t, g \rangle.$$

Let  $m^* = \max(m, m')$ . Let  $M_1^*(m^*)$ ,  $v^*(m^*)$  and  $H^*(m^*)$  be some terms such that  $\sup_{t \in B_{m, m'}(0, 1)} \|\nu_{q_n, \ell}^*(u_t^*)\|_\infty \leq M_1^*(m^*)$ ,  $\sup_{t \in B_{m, m'}(0, 1)} \text{Var}(\nu_{q_n, \ell}^*(u_t^*)) \leq v^*(m)$  and lastly  $\mathbb{E}(\sup_{t \in B_{m, m'}(0, 1)} |\nu_{n, 1}^*(u_t^*)|) \leq H^*(m^*)$ . According to Lemma 5.2 we take

$$(H^*(m^*))^2 = \frac{2\Delta(m^*)}{n}, \quad M_1^*(m^*) = \sqrt{\Delta(m^*)} \text{ and } v^*(m^*) = \frac{2\sqrt{\Delta_2(m^*, f_Z)}}{2\pi q_n},$$

where

$$(5.25) \quad \Delta_2(m, f_Z) = \int_{-\pi m}^{\pi m} \int_{-\pi m}^{\pi m} \frac{|f_Z^*(x - y)|^2}{|f_\varepsilon^*(x)f_\varepsilon^*(y)|^2} dx dy.$$

From the definition of  $T_{n, 1}^*(m, m')$ , by taking  $p_1(m, m') = 2(1 + 2\xi^2)(H^*)^2(m^*)$ , we get

$$(5.26) \quad \mathbb{E}(T_{n, 1}^*(m, m')) \leq \mathbb{E}\left[\sup_{t \in B_{m, m'}(0, 1)} |\nu_{n, 1}^*(u_t^*) - 2(1 + 2\xi^2)(H^*)^2(m^*)|_+\right].$$

According to the condition (5.16), we thus take

$$(5.27) \quad \begin{aligned} \text{pen}(m) &= 4ap(m, m) = 4a(2p_1(m, m) + 2p_2(m, m)) = 16ap_1(m, m) \\ &= 32a(1 + 2\xi^2)(2n^{-1}\Delta(m)) = 64a(1 + 2\xi^2)n^{-1}\Delta(m). \end{aligned}$$

where  $\xi^2$  is suitably chosen. Set  $m_2$  and  $m_3$  as defined in Lemma 5.2, and set  $m_1$  such that for  $m^* \geq m_1$ ,  $\Delta(m^*)$  satisfies (5.11). Take  $m_0 = m_1 \vee m_2 \vee m_3$ . We split the sum over  $m'$  in two parts and write

$$(5.28) \quad \sum_{m'=1}^{m_n} \mathbb{E}(T_{n, 1}^*(m, m')) = \sum_{m' | m^* \leq m_0} \mathbb{E}(T_{n, 1}^*(m, m')) + \sum_{m' | m^* \geq m_0} \mathbb{E}(T_{n, 1}^*(m, m')).$$

By applying Lemma 5.4, we get  $\mathbb{E}(T_{n, 1}^*(m, m')) \leq K[I(m^*) + II(m^*)]$ , where

$$I(m^*) = \frac{\sqrt{\Delta_2(m^*, f_Z)}}{p_n} \exp\left\{-2K_1\xi^2 \frac{\Delta(m^*)}{v^*(m^*)}\right\}, \quad II(m^*) = \frac{\Delta(m^*)}{p_n^2} \exp\left\{-2K_1\xi C(\xi)\sqrt{\frac{n}{q_n}}\right\}.$$

When  $m^* \leq m_0$ , with  $m_0$  finite, we get that, for all  $m \in \{1, \dots, m_n\}$ ,

$$\sum_{m' | m^* \leq m_0} \mathbb{E}(R_{n, 1}^*(m, m')) \leq \frac{C(m_0)}{n}.$$

We now come to the sum over  $m'$  such that  $m^* \geq m_0$ . It follows from Comte *et al.* (2006) that

$$(5.29) \quad v^*(m^*) = \frac{2\sqrt{\Delta_2(m^*, f_Z)}}{2\pi q_n} \leq 2\lambda_2^*(f_\varepsilon, \kappa_0) \frac{\Gamma_2(m^*)}{q_n},$$

with

$$(5.30) \quad \lambda_2^*(f_\varepsilon, \kappa_0) = \kappa_0^{-1} \sqrt{2\pi\lambda_1} \|f_\varepsilon^*\| \mathbf{1}_{\delta \leq 1} + \mathbf{1}_{\delta > 1}$$

where  $\lambda_1 = \lambda_1(f_\varepsilon, \kappa_0)$  is defined in (4.4) and

$$(5.31) \quad \Gamma_2(m) = (1 + (\pi m)^2)^\gamma (\pi m)^{\min((1/2-\delta/2), (1-\delta))} \exp(2\mu(\pi m)^\delta) = (\pi m)^{-(1/2-\delta/2)+} \Gamma(m).$$

By combining the left hand-side of (5.11) and (5.29), we get that, for  $m^* \geq m_0$ ,

$$I(m^*) \leq \frac{\lambda_2^*(f_\varepsilon, \kappa_0) \Gamma_2(m^*)}{n} \exp \left\{ -\frac{K_1 \xi^2 \lambda_1(f_\varepsilon, \kappa'_0)}{2\lambda_2^*(f_\varepsilon, \kappa_0)} (\pi m^*)^{(1/2-\delta/2)_+} \right\}$$

and

$$II(m^*) \leq \frac{\Delta(m^*) q_n^2}{n^2} \exp \left\{ -\frac{2K_1 \xi C(\xi)}{7} \frac{\sqrt{n}}{q_n} \right\}.$$

• Study of  $\sum_{m'|m^* \geq m_0} II(m^*)$ . According to the choices for  $v^*(m^*)$ ,  $(H^*(m^*))^2$  and  $M_1^*(m^*)$ , we have

$$\begin{aligned} \sum_{m'|m^* \geq m_0} II(m^*) &\leq \sum_{m' \in \{1, \dots, m_n\}} \frac{\Delta(m^*) q_n^2}{n^2} \exp \left\{ -\frac{2K_1 \xi C(\xi)}{7} \frac{\sqrt{n}}{q_n} \right\} \\ &= O \left[ m_n \exp \left\{ -\frac{2K_1 \xi C(\xi)}{7} \frac{\sqrt{n}}{q_n} \right\} \frac{\Delta(m_n) q_n^2}{n^2} \right]. \end{aligned}$$

Since  $\Delta(m_n)/n$  is bounded, then  $q_n = [n^c]$  with  $c$  in  $]0, 1/2[$  ensures that

$$(5.32) \quad \sum_{m'=1}^{m_n} m_n \exp \left\{ -\frac{2K_1 \xi C(\xi)}{7} \frac{\sqrt{n}}{q_n} \right\} \frac{\Delta(m_n) q_n^2}{n^2} \leq \frac{C}{n}.$$

Consequently

$$(5.33) \quad \sum_{m'|m^* \geq m_0} II^*(m^*) \leq \frac{C}{n}.$$

• Study of  $\sum_{m'|m^* \geq m_0} I(m^*)$ . Denote by  $\psi = 2\gamma + \min(1/2 - \delta/2, 1 - \delta)$ ,  $\omega = (1/2 - \delta/2)_+$ , and  $K' = K_1 \lambda_1(f_\varepsilon, \kappa'_0)/(2\lambda_2^*(f_\varepsilon, \kappa_0))$ . For  $a, b \geq 1$ , we use that

$$(5.34) \quad \begin{aligned} \max(a, b)^\psi e^{2\mu\pi^\delta \max(a, b)^\delta} e^{-K' \xi^2 \max(a, b)^\omega} &\leq (a^\psi e^{2\mu\pi^\delta a^\delta} + b^\psi e^{2\mu\pi^\delta b^\delta}) e^{-(K' \xi^2/2)(a^\omega + b^\omega)} \\ &\leq a^\psi e^{2\mu\pi^\delta a^\delta} e^{-(K' \xi^2/2)a^\omega} e^{-(K' \xi^2/2)b^\omega} + b^\psi e^{2\mu\pi^\delta b^\delta} e^{-(K' \xi^2/2)b^\omega}. \end{aligned}$$

Consequently,

$$(5.35) \quad \begin{aligned} \sum_{m'|m^* \geq m_0} I(m^*) &\leq \sum_{m'=1}^{m_n} \frac{\lambda_2^*(f_\varepsilon, \kappa_0) \Gamma_2(m^*)}{n} \exp \left\{ -\frac{K_1 \xi^2 \lambda_1(f_\varepsilon, \kappa'_0)}{2\lambda_2^*(f_\varepsilon, \kappa_0)} (\pi m^*)^{(1/2-\delta/2)_+} \right\} \\ &\leq \frac{2\lambda_2^*(f_\varepsilon, \kappa_0) \Gamma_2(m)}{n} \exp \left\{ -\frac{K' \xi^2}{2} (\pi m)^{(1/2-\delta/2)_+} \right\} \sum_{m'=1}^{m_n} \exp \left\{ -\frac{K' \xi^2}{2} (\pi m')^{(1/2-\delta/2)_+} \right\} \\ &\quad + \sum_{m'=1}^{m_n} \frac{2\lambda_2^*(f_\varepsilon, \kappa_0) \Gamma_2(m')}{n} \exp \left\{ -\frac{K' \xi^2}{2} (\pi m')^{(1/2-\delta/2)_+} \right\}. \end{aligned}$$

**Case**  $0 \leq \delta < 1/3$ . In that case, since  $\delta < (1/2 - \delta/2)_+$ , the choice  $\xi^2 = 1$  ensures that  $\Gamma_2(m) \exp\{-(K' \xi^2/2)(m)^{(1/2-\delta/2)_+}\}$  is bounded and thus the first term in (5.35) is bounded by  $C/n$ . Since  $1 \leq m \leq m_n$  with  $m_n$  such that  $\Delta(m_n)/n$  is bounded, the term  $\sum_{m'=1}^{m_n} \Gamma_2(m') \exp\{-(K'/2)(m')^{(1/2-\delta/2)_+}\}/n$  is bounded by  $C'/n$ , and hence

$$\sum_{m'|m^* \geq m_0} I(m^*) \leq \frac{C}{n}.$$

According to (5.16), the result follows by choosing  $\text{pen}(m) = 4ap(m, m) = 192a\Delta(m)/n$ .

**Case  $\delta = 1/3$ .** According to the inequality (5.34),  $\xi^2$  is such that  $2\mu\pi^\delta(m)^\delta - (K'\xi^2/2)m^\delta = -2\mu(\pi m^*)^\delta$  that is

$$\xi^2 = \frac{16\mu\pi^\delta \lambda_2^*(f_\varepsilon, \kappa_0)}{K_1 \lambda_1(f_\varepsilon, \kappa'_0)}.$$

Arguing as for the case  $0 \leq \delta < 1/3$ , this choice ensures that  $\sum_{m'|m^* \geq m_0} I(m^*) \leq C/n$ . The result follows by taking  $p(m, m') = 2(1 + 2\xi^2)\Delta(m^*)/n$ , and

$$\text{pen}(m) = 64a(1 + 2\xi^2) \frac{\Delta(m)}{n} = 64a \left( 1 + \frac{32\mu\pi^\delta \lambda_2^*(f_\varepsilon, \kappa_0)}{K_1 \lambda_1(f_\varepsilon, \kappa'_0)} \right) \frac{\Delta(m)}{n}.$$

**Case  $\delta > 1/3$ .** In that case  $\delta > (1/2 - \delta/2)_+$ . We choose  $\xi^2$  such that

$$2\mu\pi^\delta(m^*)^\delta - (K'\xi^2/2)(m^*)^\omega = -2\mu\pi^\delta(m^*)^\delta.$$

In other words

$$\xi^2 = \xi^2(m^*) = \frac{16\mu(\pi)^\delta \lambda_2^*(f_\varepsilon, \kappa_0)}{K_1 \lambda_1(f_\varepsilon, \kappa'_0)} (\pi m^*)^{\min((3\delta/2 - 1/2)_+, \delta)}.$$

Hence  $\sum_{m'|m^* \geq m_0} I(m^*) \leq C/n$ . The result follows by choosing  $p(m, m') = 2(1 + 2\xi^2(m, m'))\Delta(m)/n$ , associated to

$$\begin{aligned} \text{pen}(m) &= 64a(1 + 2\xi^2(m)) \frac{\Delta(m)}{n} \\ &= 64a \left( 1 + \frac{32\mu\pi^\delta \lambda_2^*(f_\varepsilon, \kappa_0)}{K_1 \lambda_1(f_\varepsilon, \kappa'_0)} (\pi m^*)^{\min((3\delta/2 - 1/2)_+, \delta)} \right) \frac{\Delta(m)}{n} \quad \square \end{aligned}$$

### Proof in the $\tau$ -dependent case.

We use the coupling properties recalled in Section 2.2 to build approximating variables for the  $W_i = (Z_i, X_i)$ 's. More precisely, we build variables  $W_i^*$  such that if  $n = 2p_n q_n + r_n$ ,  $0 \leq r_n < q_n$ , and  $\ell = 0, \dots, p_n - 1$

$$E_\ell = (W_{2\ell q_n + 1}, \dots, W_{(2\ell+1)q_n}), \quad F_\ell = (W_{(2\ell+1)q_n + 1}, \dots, W_{(2\ell+2)q_n}),$$

$$E_\ell^* = (W_{2\ell q_n + 1}^*, \dots, W_{(2\ell+1)q_n}^*), \quad F_\ell^* = (W_{(2\ell+1)q_n + 1}^*, \dots, W_{(2\ell+2)q_n}^*).$$

The variables  $E_\ell^*$  and  $F_\ell^*$  are such that

- $E_\ell^*$  and  $E_\ell$  are identically distributed,  $F_\ell^*$  and  $F_\ell$  are identically distributed,
- $\sum_{i=1}^{q_n} \mathbb{E}(\|W_{2\ell q_n + i} - W_{2\ell q_n + i}^*\|_{\mathbb{R}^2}) \leq q_n \tau_\infty(q_n)$ ,  $\sum_{i=1}^{q_n} \mathbb{E}(\|W_{(2\ell+1)q_n + i} - W_{(2\ell+1)q_n + i}^*\|_{\mathbb{R}^2}) \leq q_n \tau_\infty(q_n)$ ,
- $E_\ell^*$  and  $\mathcal{M}_0 \vee \sigma(E_0, E_1, \dots, E_{\ell-1}, E_0^*, E_1^*, \dots, E_{\ell-1}^*)$  are independent, and therefore independent of  $\mathcal{M}_{(\ell-1)q_n}$  and the same holds for the blocks  $F_\ell^*$ .

For the sake of simplicity we assume that  $r_n = 0$ . We denote by  $(Z_i^*, X_i^*) = W_i^*$  the new couple of variables.

As for the proof in the  $\beta$ -mixing framework, we start from (5.15) with  $R_n^*(m, \hat{m})$  defined by (5.14) and  $\text{pen}(m)$  chosen such that (5.16) holds. Next we use (5.17) and the bound  $|e^{-ixt} - e^{-ixs}| \leq |x||t - s|$ . Hence we conclude that

$$\sum_{i=1}^{q_n} \mathbb{E}(|e^{-iX_{2\ell_{q_n}+i}} - e^{-iX_{2\ell_{q_n}+i}^*}|) \leq q_n |x| \tau_{\mathbf{X},\infty}(q_n)$$

It follows that

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \in B_{m,\hat{m}}(0,1)} |\nu_{n,Z}(u_t^*) - \nu_{n,Z}^*(u_t^*)|^2 \right] &\leq \frac{1}{\pi} \int_{-\pi m_n}^{\pi m_n} \mathbb{E} |\nu_{n,X}^*(e^{ix}) - \nu_{n,X}(e^{ix})| dx \\ &\leq \frac{\tau_{\mathbf{X},\infty}(q_n)}{\pi} \int_{-\pi m_n}^{\pi m_n} \frac{|x|}{|f_\varepsilon^*(x)|^2} dx \\ (5.36) \qquad \qquad \qquad &\leq \tau_{\mathbf{X},\infty}(q_n) m_n \Delta(m_n). \end{aligned}$$

By gathering (5.15) and (5.36) we get

$$\mathbb{E} \|\tilde{g} - g\|^2 \leq \kappa_a^2 \|g - g_m^{(n)}\|^2 + 2a\kappa_a \sum_{m'=1}^{m_n} \mathbb{E}[T_n^*(m, m')] + 2\kappa_a \text{pen}(m) + 2a\kappa_a \tau_\infty(q_n) m_n \Delta(m_n).$$

Therefore we infer that, for all  $m \in \{1, \dots, m_n\}$ , (5.19) holds provided that

$$(5.37) \qquad \Delta(m_n) m_n \tau_\infty(q_n) \leq C_1/n \quad \text{and} \quad \sum_{m'=1}^{m_n} \mathbb{E}(T_n^*(m, m')) \leq C_2/n.$$

Using (5.11), we conclude that the first part of (5.37) is fulfilled as soon as

$$(5.38) \qquad m_n^{2\gamma+2-\delta} \exp\{2\mu\pi^\delta m_n^\delta\} \tau_\infty(q_n) \leq C'_1/n.$$

In order to ensure that our estimators converge, we only consider models with bounded penalty, that is  $\Delta(m_n) = O(n)$ . Therefore (5.38) requires that  $m_n \tau_\infty(q_n) \leq C'_1/n^2$ . For  $q_n = \lfloor n^c \rfloor$  and  $\tau_\infty(k) = O(n^{-1-\theta})$ , we obtain the condition

$$(5.39) \qquad m_n n^{-c(1+\theta)} = O(n^{-2}).$$

If  $f_\varepsilon$  satisfies (1.3) with  $\delta > 0$ , and if  $\theta > 3$ , one can find  $c \in ]0, 1/2[$ , such that (5.39) is satisfied. Now, if  $\delta = 0$  and  $\gamma \geq 3/2$  in (1.3) and if  $\theta > 3 + 2/(1 + 2\gamma)$ , then one can find  $c \in ]0, 1/2[$ , such that (5.39) is satisfied. These conditions ensure that (5.21) holds.

In order to prove the second part of (5.37), we proceed as for the proof of the second part of (5.20) and split  $T_n^*(m, m')$  into two terms

$$T_n^*(m, m') = (T_{n,1}^*(m, m') + T_{n,2}^*(m, m'))/2,$$

where the  $T_{n,k}^*(m, m')$ 's are defined in (5.22). We only study  $T_{n,1}^*(m, m')$  and conclude for  $T_{n,2}^*(m, m')$  analogously. As in the  $\beta$ -mixing framework, the study of  $T_{n,1}^*(m, m')$  consists in applying a concentration inequality to  $\nu_{n,1}^*(t)$  defined in (5.23) and considered as the sum of the  $p_n$  independent random variables  $\nu_{q_n,\ell}^*(t)$  defined as in (5.24). Once again, set  $m^* = \max(m, m')$ , and denote by  $M_1^*(m^*)$ ,  $v^*(m^*)$  and  $H^*(m^*)$  the terms such that

$\sup_{t \in B_{m,m'}(0,1)} \|\nu_{q_n,\ell}^*(u_t^*)\|_\infty \leq M_1^*(m^*)$ ,  $\sup_{t \in B_{m,m'}(0,1)} \text{Var}(\nu_{q_n,\ell}^*(u_t^*)) \leq v^*(m)$  and lastly  $\mathbb{E}(\sup_{t \in B_{m,m'}(0,1)} |\nu_{n,1}^*(u_t^*)|) \leq H^*(m^*)$ . According to Lemma 5.3, we take

$$(H^*(m^*))^2 = \frac{2\Delta(m^*)}{n}, \quad M_1^*(m^*) = \sqrt{\Delta(m^*)} \quad \text{and} \quad v^*(m^*) = \frac{C_{v^*} \sqrt{\Delta_2(m^*, f_Z)}}{2\pi q_n},$$

where  $\Delta_2(m, f_Z)$  is defined in (5.25) and where

$$(5.40) \quad C_{v^*} = 2 \left[ \mathbb{I}_{\delta > 0} + \frac{\sqrt{2}\pi^{3/2}(2\pi)^{3/2}}{\sqrt{3}} \sum_{k \geq 1} \tau_1(k) \mathbb{I}_{\delta=0} \right].$$

From the definition of  $T_{n,1}^*(m, m')$ , by taking  $p_1(m, m') = 2(1 + 2\xi^2)(H^*)^2(m^*)$ , we get

$$(5.41) \quad \mathbb{E}(T_{n,1}^*(m, m')) \leq \mathbb{E} \left[ \sup_{t \in B_{m,m'}(0,1)} |\nu_{n,1}^*(u_t^*) - 2(1 + 2\xi^2)(H^*)^2(m^*)| \right]_+.$$

As in the  $\beta$ -mixing framework we take  $\text{pen}(m) = 64a\Delta(m)(1 + 2\xi^2)/n$  where  $\xi^2$  is suitably chosen (see (5.41)). Set  $m_2$  and  $m_3$  as defined in Lemma 5.3, and set  $m_1$  such that for  $m^* \geq m_1$  (5.11) holds. Take  $m_0 = m_1 \vee m_2 \vee m_3$  and  $K' = K_1\lambda_1(f_\varepsilon, \kappa'_0)/(C_{v^*}\lambda_2^*(f_\varepsilon, \kappa_0))$ . The end of the proof is the same as in  $\beta$ -mixing framework, up to possible multiplicative constants.  $\square$

#### 5.4. Technical lemmas.

##### Lemma 5.1.

$$(5.42) \quad \left\| \sum_{j \in \mathbb{Z}} |u_{\varphi_{m,j}}^*|^2 \right\|_\infty \leq \Delta(m).$$

The proof of Lemma 5.1 can be found in Comte *et al.* (2006).

**Lemma 5.2.** Assume that  $\sum_{k \geq 1} \beta_1(k) < +\infty$ . Then we have

$$(5.43) \quad \sup_{t \in B_{m,m'}(0,1)} \|\nu_{q_n,\ell}^*(u_t^*)\|_\infty \leq \sqrt{\Delta(m^*)}$$

Moreover, there exist  $m_2$  and  $m_3$  such that

$$\mathbb{E} \left[ \sup_{t \in B_{m,m'}(0,1)} |\nu_{n,1}^*(u_t^*)| \right] \leq \sqrt{2\Delta(m^*)/n} \quad \text{for } m^* \geq m_2,$$

$$\text{and} \quad \sup_{t \in B_{m,m'}(0,1)} \text{Var}(\nu_{q_n,\ell}^*(u_t^*)) \leq 2\sqrt{\Delta_2(m^*, f_Z)}/(2\pi q_n) \quad \text{for } m^* \geq m_3,$$

where  $\Delta(m)$  and  $\Delta_2(m, f_Z)$  are defined by (3.5) and (5.25).

**Proof of Lemma 5.2.** Arguing as in Lemma 5.1 and by using Cauchy-Schwartz Inequality and Parseval formula, we obtain that the first term  $\sup_{t \in B_{m,m'}(0,1)} \|\nu_{q_n,\ell}^*(u_t^*)\|_\infty$  is bounded by

$$\sup_{t \in B_{m,m'}(0,1)} \|\nu_{q_n,\ell}^*(u_t^*)\|_\infty \leq \sqrt{\sum_{j \in \mathbb{Z}} \int \left| \frac{\varphi_{m^*,j}^*(x)}{f_\varepsilon^*(x)} \right|^2 dx} = \sqrt{\Delta(m^*)}.$$

Next

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \in B_{m,m'}(0,1)} \left| \nu_{n,1}^*(u_t^*) \right| \right] &= \mathbb{E} \left[ \sup_{t \in B_{m,m'}(0,1)} \left| \frac{1}{p_n q_n} \sum_{\ell=1}^{p_n} \sum_{i=1}^{q_n} u_t^*(Z_{2\ell q_n+i}^*) - \langle t, g \rangle \right| \right] \\ &\leq \sqrt{\sum_{j \in \mathbb{Z}} \text{Var}(\nu_{n,1}^*(u_{\varphi_{m^*,j}}^*))}. \end{aligned}$$

By using (5.6) we obtain

$$\begin{aligned} \sqrt{\sum_{j \in \mathbb{Z}} \text{Var}(\nu_{n,1}^*(u_{\varphi_{m^*,j}}^*))} &= \sqrt{\sum_{j \in \mathbb{Z}} \frac{1}{p_n^2} \sum_{\ell=1}^{p_n} \text{Var}(\nu_{q_n,\ell}^*(u_{\varphi_{m^*,j}}^*))} = \sqrt{\sum_{j \in \mathbb{Z}} \frac{1}{p_n^2} \sum_{\ell=1}^{p_n} \text{Var}(\nu_{q_n,\ell}(u_{\varphi_{m^*,j}}^*))} \\ &= \sqrt{\sum_{j \in \mathbb{Z}} \frac{1}{p_n} \text{Var}(\nu_{q_n,1}(u_{\varphi_{m^*,j}}^*))} = \sqrt{\frac{1}{2\pi p_n} \int_{-\pi m^*}^{\pi m^*} \frac{\mathbb{E}|\nu_{q_n,1}(e^{ix})|^2}{|f_\varepsilon^*(x)|^2} dx}. \end{aligned}$$

Now, according to (5.9) and (2.13)

$$\mathbb{E}|\nu_{q_n,1}(e^{ix})|^2 \leq \frac{1}{q_n} + \frac{2}{q_n} \sum_{k=1}^{n-1} \beta_1(k) |f_\varepsilon^*(x)|.$$

This implies that

$$\mathbb{E}^2 \left[ \sup_{t \in B_{m,m'}(0,1)} \left| \nu_{n,1}^*(u_t^*) \right| \right] \leq \frac{1}{p_n} \left( \frac{1}{q_n} \Delta(m^*) + \frac{2}{q_n} \sum_{k=1}^{n-1} \beta_1(k) \Delta_{1/2}(m^*) \right).$$

Since  $2 \sum_{k \geq 1} \beta_1(k) \Delta_{1/2}(m) \leq \Delta(m)$  for  $m$  large enough, we get that, for  $m^*$  large enough,

$$\mathbb{E}^2 \left[ \sup_{t \in B_{m,m'}(0,1)} \left| \nu_{n,1}^*(u_t^*) \right| \right] \leq 2\Delta(m^*)/n.$$

Now, for  $t \in B_{m,m'}(0,1)$  we write

$$\begin{aligned} \text{Var} \left( \frac{1}{q_n} \sum_{i=1}^{q_n} u_t^*(Z_{2\ell q_n+i}^*) \right) &= \text{Var} \left( \frac{1}{q_n} \sum_{i=1}^{q_n} u_t^*(Z_i) \right) \\ &= \frac{1}{q_n^2} \left[ \sum_{k=1}^{q_n} \text{Var}(u_t^*(Z_k)) + 2 \sum_{1 \leq k < l \leq q_n} \text{Cov}(u_t^*(Z_k), u_t^*(Z_l)) \right]. \end{aligned}$$

According to (5.5), (5.8) and (2.13) we have

$$\begin{aligned} |\text{Cov}(u_t^*(Z_k), u_t^*(Z_l))| &= \left| \int_{-\pi m^*}^{\pi m^*} \int_{-\pi m^*}^{\pi m^*} \frac{\text{Cov}(e^{ixZ_k}, e^{iyZ_l}) t^*(x) t^*(y)}{f_\varepsilon^*(x) f_\varepsilon^*(-y)} dx dy \right| \\ &= \left| \int_{-\pi m^*}^{\pi m^*} \int_{-\pi m^*}^{\pi m^*} \frac{f_\varepsilon^*(-y) \text{Cov}(e^{ixZ_k}, e^{iyX_l}) t^*(x) t^*(y)}{f_\varepsilon^*(x) f_\varepsilon^*(-y)} dx dy \right| \\ &\leq \int_{-\pi m^*}^{\pi m^*} \int_{-\pi m^*}^{\pi m^*} \frac{2\beta_1(k) |t^*(x) t^*(y)|}{|f_\varepsilon^*(x)|} dx dy. \end{aligned}$$

Hence,

$$\begin{aligned} \text{Var}\left(\frac{1}{q_n} \sum_{i=1}^{q_n} u_t^*(Z_{2\ell_{q_n}+i}^*)\right) &\leq \frac{1}{q_n} \left( \int_{-\pi m^*}^{\pi m^*} \int_{-\pi m^*}^{\pi m^*} \frac{f_Z^*(u-v)t^*(u)t^*(-v)}{f_\varepsilon(u)f_\varepsilon(-v)} dudv \right. \\ &\quad \left. + 2 \sum_{k=1}^{q_n} \beta_1(k) \int_{-\pi m^*}^{\pi m^*} \int_{-\pi m^*}^{\pi m^*} \left| \frac{t^*(u)t^*(v)}{f_\varepsilon^*(v)} \right| dudv \right). \end{aligned}$$

Following Comte *et al.* (2006) and applying Parseval's formula, the first integral is less than  $\sqrt{\Delta_2(m^*, f_Z)}/2\pi$ . For the second one, write

$$\int_{-\pi m^*}^{\pi m^*} \int_{-\pi m^*}^{\pi m^*} \left| \frac{t^*(u)t^*(v)}{f_\varepsilon^*(v)} \right| dudv \leq \sqrt{2\pi m^*} \|t^*\| \sqrt{\int |t^*(v)|^2 dv \int_{-\pi m^*}^{\pi m^*} \frac{dv}{|f_\varepsilon^*(v)|^2}},$$

that is

$$\int_{-\pi m^*}^{\pi m^*} \int_{-\pi m^*}^{\pi m^*} \left| \frac{t^*(u)t^*(v)}{f_\varepsilon^*(v)} \right| dudv \leq (2\pi)^2 \sqrt{m^* \Delta(m^*)}.$$

Using that  $\gamma > 1/2$  if  $\delta = 0$ , we get that  $\sqrt{m^* \Delta(m^*)} = o_m(\sqrt{\Delta_2(m^*, f_Z)})$  and hence the result follows for  $m$  large enough.  $\square$

**Lemma 5.3.** Assume that  $\sum_{k \geq 1} \tau_1(k) < +\infty$ . Assume either that

- (1)  $\delta = 0$ ,  $\gamma \geq 3/2$  in (1.3)
- (2) or  $\delta > 0$  in (1.3).

Then we have

$$(5.44) \quad \sup_{t \in B_{m,m'}(0,1)} \|\nu_{q_n, \ell}^*(u_t^*)\|_\infty \leq \sqrt{\Delta(m^*)}$$

Moreover, there exist  $m_2$  and  $m_3$  such that

$$\mathbb{E}\left[\sup_{t \in B_{m,m'}(0,1)} |\nu_{n,1}^*(u_t^*)|\right] \leq \sqrt{2\Delta(m^*)/n} \text{ for } m^* \geq m_2,$$

$$\text{and } \sup_{t \in B_{m,m'}(0,1)} \text{Var}(\nu_{q_n, \ell}^*(u_t^*)) \leq C_{v^*} \sqrt{\Delta_2(m^*, f_Z)}/(2\pi q_n) \text{ for } m^* \geq m_3,$$

where  $\Delta(m)$  and  $\Delta_2(m, f_Z)$  are defined by (3.5) and (5.25) and where  $C_{v^*}$  is defined in (5.40).

**Proof of Lemma 5.3.** The proof of (5.44) is the same as the proof of (5.43). Next, again as for the proof of Lemma 5.2

$$\mathbb{E}\left[\sup_{t \in B_{m,m'}(0,1)} |\nu_{n,1}^*(u_t^*)|\right] \leq \sqrt{\sum_{j \in \mathbb{Z}} \text{Var}(\nu_{n,1}^*(u_{\varphi_{m^*,j}}^*))}$$

with

$$\sqrt{\sum_{j \in \mathbb{Z}} \text{Var}(\nu_{n,1}^*(u_{\varphi_{m^*,j}}^*))} = \sqrt{\frac{1}{2\pi p_n} \int_{-\pi m^*}^{\pi m^*} \frac{\mathbb{E}|\nu_{q_n,1}(e^{ix})|^2}{|f_\varepsilon^*(x)|^2} dx}.$$



Now, according to (5.9) and (2.14)

$$\mathbb{E}|\nu_{q_n,1}(e^{ix})|^2 \leq \frac{1}{q_n} + \frac{1}{q_n} \sum_{k=1}^{n-1} \tau_1(k)|x||f_\varepsilon^*(x)|.$$

This implies that

$$\mathbb{E}^2 \left[ \sup_{t \in B_{m,m'}(0,1)} \left| \nu_{n,1}^*(u_t^*) \right| \right] \leq \frac{1}{p_n} \left( \frac{1}{q_n} \Delta(m^*) + \frac{2\pi}{q_n} \sum_{k=1}^{n-1} \tau_1(k) m \Delta_{1/2}(m^*) \right).$$

Since  $2\pi \sum_{k \geq 1} \tau_1(k) m \Delta_{1/2}(m) \leq \Delta(m)$  for  $m$  large enough, we get that for  $m^*$  large enough

$$\mathbb{E}^2 \left[ \sup_{t \in B_{m,m'}(0,1)} \left| \nu_{n,1}^*(u_t^*) \right| \right] \leq 2\Delta(m^*)/n.$$

Now, for  $t \in B_{m,m'}(0,1)$  we write

$$\begin{aligned} \text{Var} \left( \frac{1}{q_n} \sum_{i=1}^{q_n} u_t^*(Z_{2\ell q_n+i}^*) \right) &= \text{Var} \left( \frac{1}{q_n} \sum_{i=1}^{q_n} u_t^*(Z_i) \right) \\ &= \frac{1}{q_n^2} \left[ \sum_{k=1}^{q_n} \text{Var}(u_t^*(Z_k)) + 2 \sum_{1 \leq k < l \leq q_n} \text{Cov}(u_t^*(Z_k), u_t^*(Z_l)) \right]. \end{aligned}$$

According to (5.5), (5.8) and (2.14) and by applying the same arguments as for the proof of Lemma 5.2 we have

$$\begin{aligned} |\text{Cov}(u_t^*(Z_k), u_t^*(Z_l))| &= \left| \int_{-\pi m^*}^{\pi m^*} \int_{-\pi m^*}^{\pi m^*} \frac{f_\varepsilon^*(-y) \text{Cov}(e^{ixZ_k}, e^{iyX_l}) t^*(x) t^*(y)}{f_\varepsilon^*(x) f_\varepsilon^*(-y)} dx dy \right| \\ &\leq \int_{-\pi m^*}^{\pi m^*} \int_{-\pi m^*}^{\pi m^*} \frac{|y| \tau_1(k) |t^*(x) t^*(y)|}{|f_\varepsilon^*(x)|} dx dy. \end{aligned}$$

Hence,

$$\begin{aligned} \text{Var} \left( \frac{1}{q_n} \sum_{i=1}^{q_n} u_t^*(Z_{2\ell q_n+i}^*) \right) &\leq \frac{1}{q_n} \left( \int_{-\pi m^*}^{\pi m^*} \int_{-\pi m^*}^{\pi m^*} \frac{f_Z^*(u-v) t^*(u) t^*(-v)}{f_\varepsilon^*(u) f_\varepsilon^*(-v)} dudv \right. \\ &\quad \left. + 2 \sum_{k=1}^{q_n} \tau_1(k) \int_{-\pi m^*}^{\pi m^*} \int_{-\pi m^*}^{\pi m^*} \left| \frac{ut^*(u) t^*(v)}{f_\varepsilon^*(v)} \right| dudv \right). \end{aligned}$$

Once again the first integral is less than  $\sqrt{\Delta_2(m^*, f_Z)}/2\pi$ . For the second one, write

$$\int_{-\pi m^*}^{\pi m^*} \int_{-\pi m^*}^{\pi m^*} \left| \frac{ut^*(u) t^*(v)}{f_\varepsilon^*(v)} \right| dudv \leq \frac{\sqrt{2}\pi^{3/2}}{\sqrt{3}} (m^*)^{3/2} \|t^*\| \sqrt{\int |t^*(v)|^2 dv \int_{-\pi m^*}^{\pi m^*} \frac{dv}{|f_\varepsilon^*(v)|^2}},$$

that is

$$\int_{-\pi m^*}^{\pi m^*} \int_{-\pi m^*}^{\pi m^*} \left| \frac{t^*(u) t^*(v)}{f_\varepsilon^*(v)} \right| dudv \leq \frac{\sqrt{2}\pi^{3/2}}{\sqrt{3}} (2\pi)^{3/2} \sqrt{(m^*)^3 \Delta(m^*)}.$$

If  $\delta > 0$ , then  $\sqrt{(m^*)^3 \Delta(m^*)} = o_m \sqrt{\Delta_2(m^*, f_Z)}$ . If  $\gamma > 3/2$  and  $\delta = 0$ , we get that  $\sqrt{(m^*)^3 \Delta(m^*)} = o_m \sqrt{\Delta_2(m^*, f_Z)}$ . Lastly, if  $\gamma = 3/2$  and  $\delta = 0$ , we get that  $\sqrt{(m^*)^3 \Delta(m^*)} \leq \sqrt{\Delta_2(m^*, f_Z)}$  and the result follows for  $m$  large enough.  $\square$

**Lemma 5.4.** *Let  $Y_1, \dots, Y_n$  be independent random variables and let  $\mathcal{F}$  be a countable class of uniformly bounded measurable functions. Then for  $\xi^2 > 0$*

$$\mathbb{E} \left[ \sup_{f \in \mathcal{F}} |\nu_{n,Y}(f)|^2 - 2(1 + 2\xi^2)H^2 \right]_+ \leq \frac{4}{K_1} \left( \frac{v}{n} e^{-K_1 \xi^2 \frac{nH^2}{v}} + \frac{98M_1^2}{K_1 n^2 C^2(\xi^2)} e^{-\frac{2K_1 C(\xi) \xi}{7\sqrt{2}} \frac{nH}{M_1}} \right),$$

with  $C(\xi) = \sqrt{1 + \xi^2} - 1$ ,  $K_1 = 1/6$ , and

$$\sup_{f \in \mathcal{F}} \|f\|_\infty \leq M_1, \quad \mathbb{E} \left[ \sup_{f \in \mathcal{F}} |\nu_{n,Y}(f)| \right] \leq H, \quad \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{k=1}^n \text{Var}(f(Y_k)) \leq v.$$

This inequality comes from a concentration Inequality in Klein and Rio (2005) and arguments that can be found in Birgé and Massart (1998). Usual density arguments show that this result can be applied to the class of functions  $\mathcal{F} = B_{m,m'}(0, 1)$ .

**Proof of Proposition 2.1.** To prove (1), let for  $t > 0$ ,  $Y_t^* = \eta_t \sigma_t^*$ . Note that the sequence  $((Y_t^*, \sigma_t^*))_{t \geq 1}$  is distributed as  $((Y_t, \sigma_t))_{t \geq 1}$  and independent of  $\mathcal{M}_i = \sigma(\sigma_j, Y_j, 0 \leq j \leq i)$ . Hence, by the coupling properties of  $\tau$  (see (2.12)), we have that, for  $n + i \leq i_1 < \dots < i_l$ ,

$$\tau(\mathcal{M}_i, (Y_{i_1}^2, \sigma_{i_1}^2), \dots, (Y_{i_l}^2, \sigma_{i_l}^2)) \leq \frac{1}{l} \sum_{j=1}^l \|(Y_{i_j}^2, \sigma_{i_j}^2) - ((Y_{i_j}^*)^2, (\sigma_{i_j}^*)^2)\|_{\mathbb{R}^2} \leq \delta_n,$$

and (1) follows.

To prove (2), define the function  $f_\epsilon(x) = \ln(x) \mathbb{I}_{x > \epsilon} + 2 \ln(\epsilon) \mathbb{I}_{x \leq \epsilon}$  and the function  $g_\epsilon(x) = \ln(x) - f_\epsilon(x)$ . Clearly, for any  $\epsilon > 0$  and any  $n + i \leq i_1 < \dots < i_l$ , we have

$$(5.45) \quad \tau(\mathcal{M}_i, (Z_{i_1}, X_{i_1}), \dots, (Z_{i_l}, X_{i_l})) \leq 2\mathbb{E}(|g_\epsilon(Y_0^2)| + |g_\epsilon(\sigma_0^2)|) \\ + \tau(\mathcal{M}_i, (f_\epsilon(Y_{i_1}^2), f_\epsilon(\sigma_{i_1}^2)), \dots, (f_\epsilon(Y_{i_l}^2), f_\epsilon(\sigma_{i_l}^2)))$$

For  $0 < \epsilon < 1$ , the function  $f_\epsilon$  is  $1/\epsilon$ -Lipschitz. Hence, applying (1),

$$\tau(\mathcal{M}_i, (f_\epsilon(Y_{i_1}^2), f_\epsilon(\sigma_{i_1}^2)), \dots, (f_\epsilon(Y_{i_l}^2), f_\epsilon(\sigma_{i_l}^2))) \leq \frac{\delta_n}{\epsilon}.$$

Since  $\max(f_{\sigma^2}(x), f_{Y^2}(x)) \leq C |\ln(x)|^\alpha x^{-\rho}$  in a neighborhood of 0, we infer that for small enough  $\epsilon$ ,

$$\mathbb{E}(|g_\epsilon(Y_0^2)| + |g_\epsilon(\sigma_0^2)|) \leq K_1 \epsilon^{1-\rho} |\ln(\epsilon)|^{1+\alpha},$$

for  $K_1$  a positive constant. From (5.45), we infer that there exists a positive constant  $K_2$  such that, for small enough  $\epsilon$ ,

$$\tau(\mathcal{M}_i, (Z_{i_1}, X_{i_1}), \dots, (Z_{i_l}, X_{i_l})) \leq K_2 \left( \frac{\delta_n}{\epsilon} + \epsilon^{1-\rho} |\ln(\epsilon)|^{1+\alpha} \right).$$

The result follows by taking  $\epsilon = (\delta_n)^{1/(2-\rho)} |\ln(\delta_n)|^{-(1+\alpha)/(2-\rho)}$ .

Now, we go back to the model (2.5). If  $\sum_{j=1}^{\infty} a_j < 1$ , the unique stationary solution to (2.5) is given by Giraitis *et al.* (2000):

$$\sigma_t^2 = a + a \sum_{\ell=1}^{\infty} \sum_{j_1, \dots, j_{\ell}=1}^{\infty} a_{j_1} \dots a_{j_{\ell}} \eta_{t-j_1}^2 \dots \eta_{t-(j_1+\dots+j_{\ell})}^2.$$

for any  $1 \leq k \leq n$ , let

$$\sigma_t^2(k, n) = a + a \sum_{\ell=1}^{[n/k]} \sum_{j_1, \dots, j_{\ell}=1}^k a_{j_1} \dots a_{j_{\ell}} \eta_{t-j_1}^2 \dots \eta_{t-(j_1+\dots+j_{\ell})}^2.$$

Clearly

$$\mathbb{E}(|\sigma_n^2 - (\sigma_n^*)^2|) \leq 2\mathbb{E}(|\sigma_0^2 - \sigma_0^2(k, n)|).$$

Now

$$\mathbb{E}(|\sigma_0^2 - \sigma_0^2(k, n)|) \leq \left( \sum_{l=[n/k]+1}^{\infty} c^l + \sum_{l=1}^{\infty} c^{l-1} \sum_{j>k} a_j \right).$$

This being true for any  $1 \leq k \leq n$ , the proof of Proposition 2.1 is complete.

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